

QUASI-ISOMETRIES OF NILPOTENT GROUPS

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ABSTRACT. We prove that quasi-isometric finitely generated, torsion free nilpotent groups have isomorphic Mal'cev completions.

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1. INTRODUCTION

Since the groundbreaking work of Gromov [16], geometric group theory has been the subject of intense research efforts. In the present paper we present a contribution to the "coarse" geometric theory of nilpotent groups, giving a complete classification of finitely generated, torsion free nilpotent groups up to quasi-isometry.

Let $(X_i, d_i), i = 1, 2$ be metric spaces. Recall that X_1 and X_2 are called *quasi-isometric* if there is a map (not necessarily continuous) $f: X_1 \rightarrow X_2$ and $C, D \in \mathbb{R}_+$ such that for all $x, y \in X_1$,

$$C^{-1} \cdot d_1(x, y) - D \leq d_2(f(x), f(y)) \leq C \cdot d_1(x, y) + D,$$

and if further $\sup_{y \in X_2} \inf_{x \in X_1} d_2(y, f(x)) < +\infty$.

Given finitely generated groups Γ, Λ , we say that Γ and Λ are quasi-isometric if they admit (finite) generating sets for which the Cayley graphs are quasi-isometric. (Equivalently, the Cayley graphs are quasi-isometric for any finite generating sets.) This defines an equivalence relation among finitely generated groups. More generally, one may consider quasi-isometries

of compactly generated, locally compact groups with given word metrics; see for instance the recent book [10] for definitions.

If Γ is a finitely generated, torsion-free nilpotent group, it is well known (see section 2 below for details and references) that Γ embeds in a connected, simply connected, nilpotent Lie group G as a cocompact lattice. This G is unique. The existence and uniqueness of such an embedding was established by Mal'cev in [23]; the ambient group G is called the *Mal'cev completion* of Γ and is also denoted $G = \Gamma \otimes \mathbb{R}$.

Since any two cocompact lattices in the same ambient locally compact group are mutually quasi-isometric, it follows that any two finitely generated, torsion-free nilpotent groups with the same Mal'cev completion are in particular quasi-isometric. One may then ask whether this gives a complete characterization of quasi-isometry among finitely generated, torsion-free nilpotent groups, that is, given two quasi-isometric such groups Γ, Λ , does it follow that $\Gamma \otimes \mathbb{R} \cong \Lambda \otimes \mathbb{R}$? below we settle this in the affirmative:

Theorem A (quasi-isometry invariance of Mal'cev completion). *Let Γ, Λ be finitely generated, torsion free nilpotent groups. If Γ and Λ are quasi-isometric, then the Mal'cev completion of Γ is isomorphic to that of Λ , that is, $\Gamma \otimes \mathbb{R} \cong \Lambda \otimes \mathbb{R}$.*

1.1. A brief survey of previous work. Pansu proved an important partial result in [27]: given a nilpotent Lie algebra \mathfrak{g} the associated *graded Lie algebra* \mathfrak{g}_{gr} is defined by

$$\mathfrak{g}_{\text{gr}} := \bigoplus_{i=1}^{\text{cl}(\mathfrak{g})} \mathfrak{g}_{[i]} / \mathfrak{g}_{[i+1]},$$

where $(\mathfrak{g}_{[i]})_i$ is the lower central series in \mathfrak{g} (see Section 2 for definitions). The Lie bracket in \mathfrak{g}_{gr} of $\bar{\xi} \in \mathfrak{g}_{[i]} / \mathfrak{g}_{[i+1]}$ and $\bar{\eta} \in \mathfrak{g}_{[j]} / \mathfrak{g}_{[j+1]}$ is defined as the projection onto $\mathfrak{g}_{[i+j]} / \mathfrak{g}_{[i+j+1]}$ of $[\xi, \eta] \in \mathfrak{g}$, where ξ, η are representatives of $\bar{\xi}$ respectively $\bar{\eta}$. (It is easy to see that this is well-defined.) Thus the graded Lie algebra \mathfrak{g}_{gr} associated with \mathfrak{g} forgets all "higher degree" structure in the Lie bracket.

Pansu proved in [26] that the asymptotic cone of a connected simply connected nilpotent Lie group G identifies naturally with the graded Lie algebra associated with \mathfrak{g} . Since the asymptotic cone is a quasi-isometry invariant, the following result then follows:

Theorem (Pansu [27]). *Let Γ, Λ be finitely generated, quasi-isometric, torsion-free nilpotent groups with Mal'cev completions $G := \Gamma \otimes \mathbb{R}$ respectively $H := \Lambda \otimes \mathbb{R}$. Then $\mathfrak{g}_{\text{gr}} \cong \mathfrak{h}_{\text{gr}}$.*

In [30], Shalom, using a dynamic characterization of quasi-isometry, introduced an entirely new approach to the problem, based on a notion of cohomological induction through a measure equivalence. Specifically, Shalom showed that for finitely generated *amenable* groups, quasi-isometry is equivalent to *uniform measure equivalence* [16]: two locally compact, second countable unimodular groups G, H are called measure equivalent if they admit essentially free, commuting actions on a standard measure space (Ω, μ) , preserving the measure and such that there are measurable fundamental domains Y, X for the G - respectively H -actions with $\mu(X), \mu(Y) < +\infty$. The space (Ω, μ) is called a (G, H) -coupling. Given a G, H -coupling, one can construct cocycles $G \times X \rightarrow H$ and $H \times Y \rightarrow G$ (see Section 6 for precise definitions), and we say that the measure equivalence is *uniform* if these are (essentially) uniformly bounded on $K \times X$ for every compact subset K of G , and similarly for H . Shalom showed that, for uniformly measure equivalent countable discrete groups Γ, Λ , one can induce mutually

inverse maps

$$H^n(\Gamma, L^2X) \rightleftharpoons H^n(\Lambda, L^2Y)$$

in cohomology, and that these are continuous. Hence the reduced cohomology spaces are isomorphic: $\underline{H}^n(\Gamma, L^2X) \cong \underline{H}^n(\Lambda, L^2Y)$. By a well known result of Delorme [12] (see also the approach in [30]), it follows that $\underline{H}^n(\Gamma, L_0^2X) = \underline{H}^n(\Lambda, L_0^2Y) = 0$ where L_0^2X denotes the orthogonal complement in L^2X of the constant function $x \mapsto 1$, and analogously for L_0^2Y . From these results Shalom then deduces the following theorem:

Theorem (Shalom [30]). *Let Γ, Λ be quasi-isometric, finitely generated, torsion-free nilpotent groups. Then for all $n \in \mathbb{N}$,*

$$\beta^n(\Gamma) := \dim_{\mathbb{R}} H^n(\Gamma, \mathbb{R}) = \dim_{\mathbb{R}} H^n(\Lambda, \mathbb{R}) = \beta^n(\Lambda).$$

We mention also that Sauer [29], analyzing carefully the construction of Shalom and putting it in a functorial framework, was able to prove that the isomorphism in cohomology is consistent with cup products. (Sauer also constructed a dual isomorphism in homology, consistent with cap products.) Thus, for finitely generated, torsion-free nilpotent groups, a quasi-isometry induces an isomorphism of the *real* cohomology *rings*. Finally, we also want to mention, without going into details, the two recent papers [2, 9]

Observe that the $n = 1$ case of Shalom's theorem is already contained in Pansu's theorem, whereas the higher order cases are not: indeed, Shalom provided in [30] an example in which the Mal'cev completions of two finitely generated, torsion-free nilpotent groups can be distinguished by their second Betti number, but not by the associated graded Lie algebras.

However, as pointed out already by Shalom in [30], the result is important conceptually as well, even (and maybe even *especially*) in the case $n = 1$, providing very explicitly an isomorphism of the (tensor products with \mathbb{R} of the) abelianizations via duality. Further, Shalom's theorem, when combined with other results from [30] also gives a proof [30, Theorem 4.3.6] that any group Γ which is quasi-isometric to the integers, is virtually isomorphic to the integers; the point is that Shalom's methods do not rely on Gromov's Polynomial Growth Theorem. Our proof of Theorem A directly extends this conceptual basis. We now present an overview of the new tools needed, as well as the overall structure of our proof.

1.2. Overview of the proof of Theorem A. Analogously with [30], we deduce Theorem A from a rigidity result about uniform measure equivalence:

Theorem B. *Let G, H be connected, simply connected, nilpotent Lie groups. If G and H are uniformly measure equivalent, then they are isomorphic as Lie groups.*

Proof of Theorem A from Theorem B. Let Γ, Λ be finitely generated, torsion free nilpotent groups, and write $G := \Gamma \otimes \mathbb{R}, H := \Lambda \otimes \mathbb{R}$. Suppose that Γ, Λ are quasi-isometric. Then by [30, Theorem 2.1.2] there is a uniform (Γ, Λ) -coupling (Ω_0, μ_0) , which is ergodic, and such that Γ, Λ act freely. Indeed, we may take any ergodic uniform coupling Ω_{00} and then replace it by $\Omega_{00} \times Z$ where $\Gamma \times \Lambda \curvearrowright Z := \{0, 1\}^{\Gamma \times \Lambda}$ is the Bernoulli shift action of $\Gamma \times \Lambda$.

Now we simply put $\Omega := (G/\Gamma) \times (H/\Lambda) \times \Omega_0$ with the obvious cocycle action; it is clear that this is a uniform, ergodic (G, H) -coupling, whence Theorem A now follows directly from Theorem B. \square

The main new ingredient in the proof of Theorem B is the use of *polynomial maps* on groups: a map $\varphi: G \rightarrow H$ of groups is called a polynomial map (of degree at most $d \in \mathbb{N}_0$) if there is a $d \in \mathbb{N}_0$ such that for all $g_1, \dots, g_{d+1} \in G$, the map $(\mathfrak{c}_{g_1} \circ \dots \circ \mathfrak{c}_{g_{d+1}})(\varphi): G \rightarrow H$

equals the constant map $g \mapsto \mathbb{1}$, where the (right-)differential operator $\mathfrak{G}_g, g \in G$ is defined by $(\mathfrak{G}_g \varphi)(s) = \varphi(s)^{-1} \varphi(sg), s \in G$. A direct computation shows that the polynomial maps of degree one are precisely the affine homomorphisms (an affine homomorphism is, by definition, a homomorphism post-multiplied with a (constant) element in H).

In [21] the main result is that when the target group H above is nilpotent, the pointwise product of two polynomial maps is again a polynomial map. This extends the fact that the pointwise product of two homomorphisms into an abelian group is again a homomorphism. This and related properties of polynomial maps gives rise to the point of view that, in what can be called "higher order Fourier analysis", certain classes of polynomial maps between nilpotent groups (or more general *nilspaces*, a class of spaces containing nilmanifolds), play the rôle of homomorphisms between abelian groups in ordinary ("linear") Fourier analysis [31, 32].

Let us consider the simplest possible case of Shalom's theorem: suppose we have connected, simply connected abelian Lie groups $G := \mathbb{R}^n$ and $H := \mathbb{R}^m$, and suppose that G and H are uniformly measure equivalent. (Strictly speaking, Shalom's theorem is stated in [30] only for discrete groups, but this is, as noted therein, not a substantial restriction.) Shalom's theorem then provides a chain of isomorphisms (where X, Y denote the relevant fundamental domains given by the measure equivalence, see above)

$$(1.1) \quad \Psi: H^1(G, \mathbb{R}) \xrightarrow{\cong} \underline{H}^1(G, L^2 X) \xrightarrow{\cong} \underline{H}^1(H, L^2 Y) \xrightarrow{\cong} H^1(H, \mathbb{R}),$$

where $H^1(-, -)$ denotes the (first) *continuous cohomology* of locally compact groups. We refer to Section 4, and references therein, for detailed definitions. Suffice to say that $H^1(G, \mathbb{R})$ coincides with the group $\text{hom}(G, \mathbb{R})$ of continuous homomorphisms $G \rightarrow \mathbb{R}$. Hence, by the duality $\text{hom}(\text{hom}(G, \mathbb{R}), \mathbb{R}) \cong G$ we get an isomorphism $H \xrightarrow{\cong} G$, dual to (1.1). In particular $n = m$. Using the cocycle $\tilde{\omega}_G: H \times Y \rightarrow G$ associated with the measure equivalence (see Section 3 for details) the isomorphism can be described in cohomology (on the level of inhomogeneous cocycles) by

$$(\Psi\eta)(h) = \int_Y \eta(\tilde{\omega}_G(h, y)) d\mu_Y(y), \quad \eta \in H^1(G, \mathbb{R}).$$

Inspired by the use of polynomial maps in [21, 31], an ideal way to extend Shalom's proof to cover nilpotent groups would be to prove a chain of isomorphisms as in (1.1), but replacing the real coefficients on either extreme by a free nilpotent Lie group, and the L^2 -spaces in the middle by appropriate "nilpotent" analogues. In practice, however, to take advantage of established methods, we will define instead for each $d \in \mathbb{N}$ a cohomology theory $H_{(d)}^\bullet(-, -)$, termed (continuous) *polynomial cohomology*, with the property that for any locally compact group G , the first polynomial cohomology $H_{(d)}^1(G, \mathbb{R})$ is exactly the group $\text{Pol}_d(G)/\text{Pol}_{d-1}(G)$ of continuous polynomial maps $G \rightarrow \mathbb{R}$ of degree at most d , modulo those of degree at most $d-1$. In Section 4 we construct this "polynomial" cohomology theory, and in Section 7 we relate it to polynomial maps on groups, as well as prove several auxiliary results; notably we give an explicit description of polynomial maps on (torsion-free) nilpotent groups in terms of Mal'cev bases.

In Section 5 we prove an auxiliary result on the cohomology of nilpotent groups. In [30] a key part of the argument is the fact that finitely generated, torsion free nilpotent groups have *property H_T* : The reduced cohomology $\underline{H}^1(\Gamma, \mathcal{H}) = \underline{H}^1(\Gamma, \mathcal{H}^\Gamma)$ for any unitary Hilbert Γ -module \mathcal{H} . As alluded to above this is originally due to Delorme [12]. The aim of Section

5 is to generalize this to a wider class of modules (so-called "poly-Hilbert" modules, see Definition 5.10 for details) fitting into the framework of polynomial cohomology. Theorem 5.15 is a key technical foundation of the methods presented in this paper.

In Section 6 we construct an isomorphism in polynomial cohomology, analogous to the middle isomorphism in (1.1) to the setting of polynomial cohomology. We do this by proving a Reciprocity Theorem generalizing that in [25]. Our Theorem 6.4 gives an isomorphism

$$(1.2) \quad H_{(d)}^n(G, L^2 X) \xrightarrow{\cong} H^n(H, L^2(Y, \text{Pol}_{d-1}(G))) .$$

Heuristically, one would expect by induction then to identify $\text{Pol}_{d-1}(G)$ with $\text{Pol}_{d-1}(H)$, in a natural manner so as to get as explicit a map on the level of cohomology as possible. This will take up the final part of the paper, after the introduction of several technical tools.

As a byproduct of the methods of Section 6 we also give a description of polynomial cohomology in terms of ordinary cohomology (see Proposition 6.8).

The family of all $H_{(d)}^1(G, \mathbb{R})$, $d \in \mathbb{N}_0$ (where by convention $H_{(0)}^1(G, \mathbb{R}) = \mathbb{R}$) identifies with the group $\text{Pol}(G)$ of all continuous, real-valued polynomials on G , and this will act as a sort of "total" higher order dual space. However, in order to capture the group structure on G , and not just the ranks of the successive subquotients $G_{[i]}/G_{[i+1]}$ in the lower central series of G , we need to take into account also certain structure maps on $\text{Pol}(G)$. Namely, $\text{Pol}(G)$ is an algebra, with pointwise multiplication of polynomials as the algebra multiplication. Further, the multiplication map $m: G \times G \rightarrow G$ induces a pull-back map $m^*: \text{Pol}(G) \rightarrow \text{Pol}(G \times G)$. In Section 8 we give precise constructions of these structure maps, and show that a map $\Phi: \text{Pol}(G) \rightarrow \text{Pol}(H)$ (for G, H connected, simply connected nilpotent Lie groups, say) which preserves the structure maps, induces a homomorphism $\varphi: H \rightarrow G$ such that the pull-back $\varphi^*: \text{Pol}(G) \rightarrow \text{Pol}(H)$ coincides with Φ .

Section 9 is devoted to the identification of the right-hand side of (1.2) with $H_{(d)}^1(H, \mathbb{R})$. Finally, in Section 10 we finish the proof of Theorem B by proving that, for uniformly measure equivalent, connected, simply connected nilpotent Lie groups G, H , the isomorphisms $H_{(d)}^1(G, \mathbb{R}) \xrightarrow{\cong} H_{(d)}^1(H, \mathbb{R})$, $d \in \mathbb{N}$ constructed in Section 9, can be glued together to an isomorphism $\text{Pol}(G) \xrightarrow{\cong} \text{Pol}(H)$ respecting all the relevant structure maps from Section 8, and so induces an isomorphism of groups.

1.3. Notation and conventions.

Topological vector spaces. In this paper all (topological) vector spaces will be real and assumed to satisfy the Hausdorff separation axiom, unless explicitly stated otherwise. By definition, a *morphism* $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ between not necessarily Hausdorff topological vector spaces \mathcal{E}, \mathcal{F} is a continuous linear map, and an *isomorphism* is a morphism such that there exists an inverse morphism $\varphi^{-1}: \mathcal{F} \rightarrow \mathcal{E}$.

Topological groups. A 'group' G will always mean simply an abstract group without any topology. We will abbreviate 'locally compact second countable (unimodular)' by 'lscs' (respectively 'lscsu'). We denote the identity element in G by $\mathbb{1}_G$, leaving out the subscript whenever possible.

Measure spaces. We refer to Section 3 for detailed statements of our terminology concerning (Borel) measure spaces. Given a measure space (X, μ) and any statement involving it, the

terminology 'for almost every (a.e.) $x \in X$ (...)' is synonymous with 'there exists a subset X_0 of X of co-null measure such that for every $x \in X_0$ (...)'

Extended natural numbers. We denote $\mathbb{Z}_* := \{-\infty\} \cup \mathbb{N}_0$ and define

$$\begin{aligned} x \dot{+} y &:= \begin{cases} x + y, & x, y \in \mathbb{N}_0 \subseteq \mathbb{Z}_* \\ -\infty, & \text{if either of } x, y = -\infty \end{cases} \\ x \dot{-} y &:= \begin{cases} x - y, & x \geq y \in \mathbb{N}_0 \subseteq \mathbb{Z}_* \\ -\infty, & x = -\infty \text{ or } x < y \in \mathbb{N}_0 \end{cases} \end{aligned}$$

We leave $x \dot{-} y$ undefined if both $x, y = -\infty$.

Multi-index notation. A *multi-index* (over I) is an element $\mathbf{d} = (d_i)_{i \in I} \in \mathbb{N}_0^I$ for some finite set I . For a Mal'cev group G (see Section 2 for the definition of Mal'cev groups; in particular this includes G csc nilpotent Lie) we denote by $\mathbf{cl}(G)$ the multi-index $(1, \dots, \text{cl}(G)) \in \mathbb{N}_0^{\text{cl}(G)}$, by $\mathbf{rk}(G)$ the multi-index $(\dim_{\mathbb{R}} \mathfrak{g}_{[i]}/\mathfrak{g}_{[i+1]})_{i=1, \dots, \text{cl}(G)} \in \mathbb{N}_0^{\text{cl}(G)}$. For any multi-index \mathbf{k} over I we denote by $\mathbb{N}_0^{\mathbf{k}}$ the disjoint union $\dot{\cup}_{i \in I} \mathbb{N}_0^{k_i}$. Finally we denote by $\mathbf{dim}(G)$ the multi-index (where we write $\mathbf{m} := \mathbf{rk}(G)$)

$$\mathbf{dim}(G) := ((1)_{j=1, \dots, m_1}, \dots, (\text{cl}(G))_{j=1, \dots, m_{\text{cl}(G)}}) \in \mathbb{N}_0^{\mathbf{m}}.$$

For any $d \in \mathbb{N}_0$ and any multi-index \mathbf{k} over I we define

$$\mathbf{D}_{d, \mathbf{k}} := \left\{ \mathbf{d} \in \mathbb{N}_0^I \mid \sum_{i \in I} k_i d_i \leq d \right\},$$

and denote by $\mathbf{D}_{d, \mathbf{k}}^-$ the subset for which equality holds. Finally, we set

$$\mathbf{B}_{\mathbf{k}} := \dot{\cup}_{i \in I} \{1, 2, \dots, k_i\},$$

and consider on this set the lexicographic ordering.

Product notation. Let G be a group. For any finite, well-ordered set I and any map $i \rightarrow G: i \mapsto g_i$ we write $\prod_{i \in I} g_i$ for the element g in G defined recursively by $g = g_{i_0} \cdot \prod_{i \in I \setminus \{i_0\}} g_i$, where i_0 is the smallest element in I .

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2. PRELIMINARIES ON NILPOTENT GROUPS

In this section we collect the necessary prerequisites concerning nilpotent groups. For general background we refer to [11, 19].

Let G be a group. Recall that the commutator of two elements $g, h \in G$ is the element $[g, h] := g^{-1}h^{-1}gh \in G$. For subgroups $H, K \subseteq G$ the commutator $[H, K]$ is the subgroup generated by all elements $[h, k], h \in H, k \in K$.

Definition 2.1 (central series). Recall that a central series $\mathcal{G} = (G_i)_{i \in \mathbb{N}}$ in a (topological) group G is a decreasing sequence of (closed) normal subgroups $G_i \trianglelefteq G$, with $G = G_1$ and such that $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \in \mathbb{N}$.

The *lower central series* of a topological group G is the (decreasing) sequence $\mathcal{G}_{min} = (G_{[i]})$ of subgroups of G defined recursively by $G_{[1]} := G$ and $G_{[i+1]} := \overline{[G, G_{[i]}]}$. Observe that each $G_{[i]}$ is a characteristic subgroup of G . Further, for any central series $\mathcal{G} = (G_i)$ on G we have, by construction, $G_{[i]} \leq G_i$.

A group G is called *nilpotent* if $G_{[d]} = \{1\}$ for some $d \in \mathbb{N}$; in this case the (*nilpotency*) *class* of G is the number $\text{cl}(G) := \min\{d \mid G_{[d]} = \{1\}\} - 1$. In the remainder of this section we recall some fundamental results about nilpotent (torsion-free) groups, and introduce the class of groups we will work with in the present paper.

Recall that for any Lie algebra \mathfrak{g} , the lower central series is defined (analogously to the definition for groups) by $\mathfrak{g}_{[i+1]} = \text{span}[\mathfrak{g}, \mathfrak{g}_{[i]}]$. Let G be a connected, simply connected (henceforth abbreviated 'csc'), nilpotent Lie group, and denote its Lie algebra \mathfrak{g} . Recall that for such G , the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a *global* diffeomorphism onto G , inducing in particular for each i diffeomorphisms $\mathfrak{g}_{[i]}/\mathfrak{g}_{[i+1]} \rightarrow G_{[i]}/G_{[i+1]}$.

A (strong) Mal'cev basis of \mathfrak{g} (wrt. the lower central series in \mathfrak{g}) is a linear basis $(X_{i,j})_{\mathbf{B}_{\text{rk}(G)}}$ of \mathfrak{g} , such that for each i , $X_{i,j} \in \mathfrak{g}_{[i]}$ for all j , and the set $\{X_{i,j}\}_j$ projects to a linear basis of $\mathfrak{g}_{[i]}/\mathfrak{g}_{[i+1]}$. Such a basis always exists. Then the map

$$(2.1) \quad \mathfrak{g} \ni \sum_{(i,j) \in \mathbf{B}_{\text{rk}(G)}} t_{i,j} X_{i,j} \mapsto \prod_{(i,j) \in \mathbf{B}_{\text{rk}(G)}} \exp(t_{i,j} X_{i,j}).$$

is a diffeomorphism as well [11, Section 1.2], and the induced coordinate system on G is called (the system of) Mal'cev coordinates. Abusing terminology, we will therefore also refer to the family $\{g_{i,j} := \exp(X_{i,j}) \mid (i,j) \in \mathbf{B}_{\text{rk}(G)}\}$ as a Mal'cev basis of G .

Given any Mal'cev basis $(X_{i,j})$ of \mathfrak{g} , for all i, j there are constants $\{c_{k,l}^{i,j,s,t}\}$ such that

$$[X_{i,j}, X_{s,t}] = \sum_{(k,l) \in \mathbf{B}_{\text{rk}(G)}} c_{k,l}^{i,j,s,t} X_{k,l}.$$

The (indexed) collection of all $c_{k,l}^{i,j,s,t}$ is called the set of structure constants of \mathfrak{g} (wrt. $(X_{i,j})$). In his groundbreaking paper [23], Mal'cev proved the following result:

Theorem 2.2 (Mal'cev). *A csc nilpotent Lie group G has a lattice if and only if it has a Mal'cev basis with rational structure constants. Furthermore, every lattice in G is cocompact.*

Any lattice in a csc nilpotent Lie group is necessarily torsion-free and finitely generated. Mal'cev proved also that the converse is true:

Theorem 2.3 (Mal'cev). *Let Γ be a finitely generated, torsion-free (discrete) nilpotent group. Then there exists a csc nilpotent Lie group G such that Γ embeds as a lattice in G . Furthermore, the embedding is unique up to natural isomorphism, that is, given any two such embeddings $i: \Gamma \rightarrow G$ and $j: \Gamma \rightarrow H$, there is an isomorphism $\psi: G \rightarrow H$ intertwining i and j .*

The csc nilpotent Lie group G in the theorem is called the Mal'cev completion of Γ , and is occasionally denoted $G := \Gamma \otimes \mathbb{R}$. For a proof of the theorem see [23], or for an alternative approach [6] (which is based on [18]). We will indicate an approach below as well.

Finally we note that any locally compact, compactly generated, totally disconnected nilpotent group contains a neighbourhood basis of the identity consisting of compact open *normal* subgroups [36]. Thus, up to quotienting out a totally disconnected compact group, such groups can be studied via discrete nilpotent groups. The techniques developed in this paper,

based on cohomology with coefficients in vector spaces, essentially do not "see" compact subgroups; this motivates the following definition:

Definition 2.4 (Mal'cev group). Let G be a locally compact, compactly generated topological group. We will say that G is a *Mal'cev group* if it satisfies any (hence all) of the following equivalent criteria:

- (i) G embeds as a closed, cocompact subgroup in a csc nilpotent Lie group,
- (ii) G has a finite length central series (G_i) , that is, $G_d = \{1\}$ for some d , such that $G_i/G_{i+1} \cong \mathbb{R}^{m_i} \times \mathbb{Z}^{m'_i}$ for all i ,
- (iii) G is a torsion free, nilpotent Lie group.

In case (ii) we can always find such a central series such that, furthermore, for each i the quotient $G_i/G_{[i]}$ is compact. It can be proved that such a central series is uniquely determined by these requirements and we will therefore refer to it as *the Mal'cev central series*.

The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear; $(iii) \Rightarrow (i)$ is due to Mal'cev in the discrete case and to Wang [35, Prop. 4.6] in general.

Definition 2.5. We will call *residually Mal'cev group* any compactly lcsc group G for which there is a continuous, injective homomorphism $\varphi: G \rightarrow N$ into a (countable) projective limit $N = \lim_{\rightarrow} N_i$ of csc nilpotent Lie groups N_i such that

- (i) the ranks of the abelianizations $N_i/[N_i, N_i]$ are uniformly bounded, and
- (ii) for every i , the induced homomorphism $\varphi_i: G \rightarrow N_i$ has closed, cocompact image.

Since the structure constants of any csc nilpotent Lie group are witnessed by any closed cocompact subgroup, the ambient csc nilpotent Lie group in the previous definition is uniquely determined by G . Generalizing the discrete case we call it the Mal'cev completion of G and denote it $G \otimes \mathbb{R}$. More generally, if G is a residually Mal'cev group, we define the Mal'cev completion $G \otimes \mathbb{R} := \lim_{\leftarrow} (G/G_i) \otimes \mathbb{R}$. For any such G we will by abuse of notation denote by \mathfrak{g} the projective limit of the Lie algebras of $(G/G_i) \otimes \mathbb{R}$; in particular for a Mal'cev group G we will abuse conventions and write \mathfrak{g} directly for the Lie algebra of $G \otimes \mathbb{R}$.

Definition 2.6 (length and rank). Let G be a residually Mal'cev group. We denote the length of the (Mal'cev-, equivalently lower-) central series by $\text{cl}(G)$, if it is finite. We denote by $\text{rk}(G)$ the *rank* of G , defined by $\text{rk}(G) := \dim_{\mathbb{R}} \mathfrak{g}/\mathfrak{g}_{[2]}$, where \mathfrak{g} is the Lie algebra of $G \otimes \mathbb{R}$. That is, we have $G_1/G_2 \cong \mathbb{R}^m \times \mathbb{Z}^n$ for some uniquely determined m, n , and we define $\text{rk}(G) = m + n$.

Given a Mal'cev group G it can be proved (see the following remark) that there is a Mal'cev basis $\{g_{i,j}\}$ of $G \otimes \mathbb{R}$ with $g_{i,j} \in G$ for all i, j . We will call such a family a Mal'cev basis of G . In general, for G a residually Mal'cev group we will call Mal'cev basis of G , any family $\{g_{i,j}\}$ of elements in G such that the images $\{\bar{g}_{i,j}\}_{(i,j) \in \mathbf{B}_{\text{rk}(G/G_{d+1})}}$ is a Mal'cev basis in G/G_{d+1} .

Remark 2.7. Letting $g_{i,j}, (i, j) \in \mathbf{B}_{\text{rk}(G)}$ denote the pre-images of the canonical basis vectors e_j wrt. some fixed isomorphisms $G_i/G_{i+1} \cong \mathbb{R}^{c'_i} \times \mathbb{Z}^{c''_i}$, we have that the $g_{i,j}$ constitute a Mal'cev basis of G , and that (setting $X_{i,j} = \log(g_{i,j})$ and) restricting appropriate coordinates $t_{i,j}$ to \mathbb{Z} , the map in (2.1) yields a "Mal'cev coordinate system" for H . In this way, the non-rigorous, heuristic definition of a (residually) Mal'cev group is that it is a (compactly generated) group which can be (residually) described, in some sense, by a Mal'cev coordinate system.

- Example 2.8** (examples of Mal'cev groups). (i) By the theorem of Mal'cev, every finitely generated torsion-free nilpotent group is a Mal'cev group. More generally, every finitely generated, residually torsion free nilpotent group is residually Mal'cev.
- (ii) By a classical result of Magnus, see e.g. [22], free groups are residually torsion free nilpotent. The class of (finitely generated) residually torsion-free nilpotent groups is known to be stable under free products, as well as certain amalgamated free products.
- (iii) Surface groups, right-angled Artin groups [13], and pure braid groups [14, 15] are residually torsion-free nilpotent.

In [8], various examples are produced to show that the *nilpotent genus*, that is, the family of nilpotent quotients, does not in general retain information about certain isomorphism invariants of residually torsion free nilpotent groups.

For any $n, d \in \mathbb{N}$ we denote by $F_n(d)$ the free nilpotent group of class d on n generators. That is, denoting by F_n the free group on n generators f_1, \dots, f_n , we set $F_n(d) := F_n/(F_n)_{[d+1]}$. This group satisfies the following universal condition: given any nilpotent group Γ of class at most d and any $g_1, \dots, g_n \in \Gamma$, there is a unique group homomorphism $\varphi: F_n(d) \rightarrow \Gamma$ such that $\varphi(f_i) = g_i, i = 1, \dots, n$. Similarly, we denote by $\mathfrak{f}_n(d)$ the free nilpotent Lie algebra of class d on n generators $X_i, i = 1, \dots, n$, and by $N_n(d)$ the associated csc (nilpotent) Lie group. Since the structure constants of $\mathfrak{f}_n(d)$ wrt. a canonical Mal'cev basis of $N_n(d)$ consisting of (a subset of all of the) commutators of the X_i are rational, such a basis determines a cocompact lattice in $N_n(d)$ by Mal'cev's theorem, and it is clear that this is precisely the free nilpotent group $F_n(d)$.

In particular, using this we can now indicate a construction of the Mal'cev completion $\Gamma \otimes \mathbb{R}$ of any finitely generated, torsion-free nilpotent group Γ . Let F be a free nilpotent group surjecting onto Γ , say $\varphi: F \rightarrow \Gamma$, and let N be the associated free nilpotent Lie group. (Note in particular that, by the above, we have *a priori* $N = F \otimes \mathbb{R}$.) Let $K := \ker \varphi$. By [11, Theorem 5.4.3] there is a unique smallest Lie group $L \leq N$ containing K , and K is cocompact in L . (Note, again, that thus we have in fact $L = K \otimes \mathbb{R}$.) The cocompactness ensures that, since Γ is torsion-free, $L \cap F = K$, and it follows from this that Γ is a cocompact lattice in the csc nilpotent Lie group N/L .

3. MEASURE EQUIVALENCE AND UNIFORM MEASURE EQUIVALENCE

In this section we fix the terminology we will use for measure spaces, in particular we make precise our use of the term 'standard space', which is maybe not entirely standard. Further, it will be technically convenient to be able to speak of *the* family of compact subsets of a standard Borel space, since the compact subsets will appear in the definition of certain function spaces on standard Borel spaces in the sequel.

Definition 3.1 (measure space terminology). By a Borel space X we mean a triple $(X, \mathcal{B}, \mathcal{T})$ where \mathcal{T} is a topology on X and \mathcal{B} is the Borel σ -algebra generated by \mathcal{T} , that is, the smallest σ -algebra containing \mathcal{T} . We will often suppress the topology when speaking of a Borel space, but emphasize here that it is part of the data.

By a measure space (X, μ) we mean simply a set X endowed with a σ -algebra, and a measure μ on this. By a measure, we mean a positive measure, taking values in $[0, \infty]$. A Borel measure space is thus a Borel space, with a measure on the Borel σ -algebra. Given a Borel measure space (X, \mathcal{B}, μ) , we will often consider implicitly also the completed σ -algebra

$\bar{\mathcal{B}}$ of \mathcal{B} with respect to μ . The word 'measurable' (e.g. measurable set, measurable map, etc.) will mean measurable with respect to the completed σ -algebra, whereas the word 'Borel' (-set, -map, etc.) pertains to \mathcal{B} .

A Borel map $\varphi: X \rightarrow Y$ of Borel spaces X, Y is called *locally bounded* if it maps relatively compact sets to relatively compact sets.

A Borel measure space (X, μ) will be called *standard* if there is a Borel isomorphism $\varphi: X \rightarrow \mathbb{R} \dot{\cup} \mathbb{Z}$ onto a closed subset of the disjoint union of \mathbb{R} with Lebesgue measure and \mathbb{Z} with counting measure, such that:

- (i) the push-forward $\varphi_*\mu$ is equivalent to the restriction of the Lebesgue+counting measure on $\mathbb{R} \dot{\cup} \mathbb{Z}$ to the range of φ , and,
- (ii) both maps φ, φ^{-1} are locally bounded.

Remark 3.2. The requirement in the definition of 'standard Borel measure space' that the isomorphisms φ, φ^{-1} be bounded, is non-standard compared to the general literature. Note however, that for σ -compact, locally compact Borel measure spaces, our terminology is equivalent to the usual one.

3.3. Measure equivalence couplings of lscu groups. Next we give the definition of measure equivalence coupling between two lscu groups that we will use in this paper. Our definition essentially coincides essentially with the one used in [4].

Definition 3.4 (couplings). Let G, H be lscu groups. A measure equivalence (ME) coupling from G to H (also called a (G, H) -coupling) is a standard Borel measure space (Ω, μ) along with:

- (i) A locally bounded Borel $G \times H$ action $\sigma: G \times H \times \Omega \rightarrow \Omega$; that is, the map σ is assumed to be a locally bounded Borel map. We write $\sigma_G: G \times \Omega \rightarrow \Omega$ respectively $\sigma_H: H \times \Omega \rightarrow \Omega$ for the restricted actions. Generally, we denote simply $\sigma(g, h)(t)$ by $(g, h).t = g.h.t = h.g.t$.
- (ii) Two measure preserving Borel isomorphisms

$$i: (G, \lambda_G) \times (Y, \mu_Y) \rightarrow (\Omega, \mu)$$

$$j: (H, \lambda_H) \times (X, \mu_X) \rightarrow (\Omega, \mu),$$

where $\mu_X(X), \mu_Y(Y) \in (0, +\infty)$ and $(X, \mu_X), (Y, \mu_Y)$ are standard Borel measure spaces.

Further, we require that all four maps i, i^{-1}, j, j^{-1} are locally bounded, and that

- (iii) all maps i, i^{-1}, j, j^{-1} are essentially G - respectively H -equivariant, i.e. for almost every $y \in Y$ and all $g_0, g \in G$, we have $i(g_0g, y) = g_0.i(g, y)$, and analogously for H .

We say that the (G, H) -coupling (Ω, μ) is *uniform* if X, Y are compact.

Remark 3.5. As implied by the name, 'measure equivalence' is indeed an equivalence relation.

Remark 3.6. Let (Ω, μ) be a (G, H) -coupling. It follows directly from the definition that the $G \times H$ -action on Ω is essentially free and preserves μ .

Example 3.7. Let Γ, Λ be finitely generated (countable discrete) amenable groups. If Γ and Λ are quasi-isometric, then by [30, Theorem 2.1.2] (see also [16, 0.2.C'_2] for the original formulation) they are uniformly measure equivalent.

Given a (G, H) -coupling (Ω, μ) , we get, by the equivariance of i, j , essentially free, measure preserving Borel actions $G \times X \rightarrow X$ and $H \times Y \rightarrow Y$. Indeed, by (iii) of the definition of

(G, H) -coupling we fix a subset $\Omega_0 \subseteq \Omega$ of co-null measure and a subset $X_0 \subseteq X$, also of co-null measure, such that the restriction $j|_1: H \times X_0 \xrightarrow{\cong} \Omega_0$ is H -equivariant, and such that $G \times H$ acts freely on Ω_0 . Then we can define the action of G on X_0 by $j|_1(H \times \{g.x\}) = g.j|_1(H \times \{x\})$; this clearly extends to an essentially free (locally bounded Borel-) action of G on X . The H -action on Y is defined analogously.

Remark 3.8. We will frequently use the well-known fact that, given any (G, H) -coupling (Ω, μ) , we can replace μ by an *ergodic* invariant measure. Observe that μ is ergodic if and only if μ_X is ergodic. By symmetry, μ is ergodic if and only if μ_Y is ergodic.

First, we note that there is an ergodic, G -invariant probability measure ν_X on X : indeed denote by $\mathcal{S} \subseteq C(X)^*$ the set of G -invariant probability measures on X . Clearly \mathcal{S} is a weak-* closed, convex set whence by the Krein-Milman Theorem [28, Theorem 3.23] there is an extreme point ν_X in \mathcal{S} . Now we can define ν as the push-forward $j_*(\lambda_H \otimes \nu_X)$. Finally, by disintegration, there is a Borel measure $\nu'_Y := (\pi_Y)_*(\nu)$ on Y and a field $y \mapsto \lambda^y$ of Borel measures on G such that $\nu = i_*(\int_Y \lambda^y d\nu'_Y)$. But then by uniqueness of the Haar measure on G it follows that $\lambda^y = c(y) \cdot \lambda_G$, where $c(y) > 0$ for almost every y . Now take $\nu_Y = c(-)^{-1} \cdot \nu'_Y$. Then (Ω, ν) satisfies the claim with (X, ν_X) and (Y, ν_Y) in place of (X, μ_X) respectively (Y, μ_Y) .

3.9. Cocycles associated with a measure equivalence coupling. Recall that, given groups G, H and a set X with an action of G , a (left-)cocycle $\omega: G \times X \rightarrow H$ is a map satisfying

$$\omega(gh, x) = \omega(g, h.x) \cdot \omega(h, x), \quad g, h \in G.$$

Let G, H be lcscu groups and let (Ω, μ) be a (G, H) -coupling. To this we associate two cocycles (see below) $\omega_H: G \times X \rightarrow H$ and $\omega_G: H \times Y \rightarrow G$, defined by

$$\begin{aligned} g.j(h, x) &= j(h\omega_H(g, x)^{-1}, g.x), \quad \text{for a.e. } x \in X \text{ and all } h \in H, \\ h.i(g, y) &= i(g\omega_G(h, y)^{-1}, h.y), \quad \text{for a.e. } y \in Y \text{ and all } g \in G. \end{aligned}$$

Remark 3.10. To see that ω_H , say, is well-defined we proceed as follows: by definition, for each $x \in X_0$, where $X_0 \subseteq X$ has co-null measure, and every $g \in G$, we have $g.j(H \times \{x\}) = j(H \times \{g.x\})$. Hence for each $h \in H$ there is a unique element $\omega_H^{(h)}(g, x) \in H$ such that $g.j(h, x) = j(h\omega_H^{(h)}(g, x)^{-1}, g.x)$. Further, by almost-equivariance of j , the map $h \mapsto \omega_H^{(h)}(g, x)$ is essentially constant. This defines ω_H on $G \times X_0$, and we extend this to $G \times X$ by setting $\omega_H: G \times (X \setminus X_0) \mapsto \mathbb{1}$.

It follows directly from the construction that $\omega_H: G \times X_0 \rightarrow H$ satisfies the cocycle identity 3.9, and so does $\omega_G: H \times Y_0 \rightarrow G$. That is, ω_H, ω_G satisfy the cocycle identity a.e. on X respectively Y . In the sequel, to avoid the notational inconvenience, we might leave out explicit choice of X_0, Y_0 when no confusion can arise from treating the cocycles as if they satisfy 3.9 everywhere.

Proposition 3.11. *The cocycles ω_H, ω_G associated with the (G, H) -coupling (Ω, μ) are (essentially) locally bounded Borel maps.*

Proof. Fix a compact set $K \subseteq H$ such that $\lambda_H(K) = 1$. Then for all $g \in G, x \in X_0$ we have for a.e. $h \in K$ that $\omega_H(g, x)^{-1} = h^{-1} \cdot (\pi_H \circ (j^{-1}))(g.j(h, x))$. For every $f \in C(H)$ we then

have,

$$(3.1) \quad f(\omega_H(g, x)^{-1}) = \int_K f(h^{-1} \cdot (\pi_H \circ (j^{-1}))(g \cdot j(h, x))) d\lambda_H(h).$$

Hence for every $f \in C(H)$ the map $(g, x) \mapsto f(\omega_H(g, x)^{-1})$ is Borel, from which it follows that $(g, x) \mapsto \omega_H(g, x)$ is Borel. Further, for $L \subseteq G$ and $Z \subseteq X$ compact sets, the set $A := \{h^{-1} \cdot (\pi_H \circ (j^{-1}))(g \cdot j(h, x)) \mid h \in K, g \in L, x \in Z\}$ is relatively compact in H , and by (3.1) it follows that $\omega_H(L \times Z)^{-1} \subseteq \overline{A}$. \square

To ease notation we also write $\tilde{\omega}_G(h, y) := \omega_G(h^{-1}, y)^{-1}$, and similarly for $\tilde{\omega}_G$. This then satisfies the cocycle identity

$$\tilde{\omega}_G(gh, y) = \tilde{\omega}_G(g, y) \cdot \tilde{\omega}_G(h, g^{-1} \cdot y).$$

Finally, we define an a.e. G -equivariant map $\chi_G: \Omega \rightarrow G$ by $\chi_G = \pi_G \circ (i^{-1})$, that is, $\chi_G(t)$ is the unique element $\chi_G(t) \in G$ such that $\chi_G(t)^{-1} \cdot t \in i(\{1\} \times Y)$. Then χ_G satisfies (and can equivalently be defined by) the identity

$$(3.2) \quad g^{-1} \cdot \chi_G(h \cdot i(g, y)) = \omega_G(h, y)^{-1}, \quad \text{for a.e. } y \in Y \text{ and all } g \in G.$$

In particular, χ_G is a (essentially) locally bounded Borel map.

4. POLYNOMIAL COHOMOLOGY OF LCSC GROUPS

In this section we recall the definition of continuous cohomology for locally compact groups. We also define, more generally, a notion of "polynomial cohomology", the "linear" (or degree one) case of which is the usual cohomology.

Definition 4.1 (continuous G -modules). Let G be a lcsc group and \mathcal{E} be a locally convex topological vector space. We say that \mathcal{E} is a *continuous G -module* if it is endowed with a representation of G on \mathcal{E} such that the action map $G \times \mathcal{E} \rightarrow \mathcal{E}$ is continuous. A morphism $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ of continuous G -modules is, by definition, a continuous, G -equivariant linear map. We say that φ is *strengthened* if there is a continuous linear map $\eta: \mathcal{F} \rightarrow \mathcal{E}$ such that $\varphi \circ \eta \circ \varphi = \varphi$.

Observation 4.2. For G a lcsc group and X a locally compact space on which G acts continuously by homeomorphisms, the space of continuous functions $C(X, \mathcal{E})$ is a continuous G -module for every continuous G -module \mathcal{E} , with respect to the standard action $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$. Recall that the topology on $C(X, \mathcal{E})$ is the projective topology generated by the restriction maps $C(X, \mathcal{E}) \rightarrow C(K, \mathcal{E})$ over all compact subsets K of X , that is, the topology of uniform convergence on compact sets. In particular, note that if X is second countable and \mathcal{E} is a Fréchet space (as will often be the case in this paper), then $C(X, \mathcal{E})$ is a Fréchet space as well.

Definition 4.3 (relative injectivity). We say that a continuous G -module \mathcal{E} is relatively injective if given any diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathcal{F}_1 \xrightarrow{u} \mathcal{F}_2 \\ & & \downarrow v \quad \swarrow \exists?w \\ & & \mathcal{E} \end{array}$$

where $u: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a strengthened injective morphism, there is a morphism $w: \mathcal{F}_2 \rightarrow \mathcal{E}$ such that the augmented diagram commutes.

Lemma 4.4 ([17, Chapter III]). *Let G be a lcsc group and \mathcal{E} be a continuous G -module. Then $C(G, \mathcal{E})$ is relatively injective. In particular, the category of continuous G -modules contains sufficiently many relatively injectives. Further, there is a strengthened injective resolution (with the usual coboundary maps [17])*

$$0 \longrightarrow \mathcal{E} \longrightarrow C(G, \mathcal{E}) \longrightarrow \cdots \longrightarrow C(G^\bullet, \mathcal{E}) \longrightarrow \cdots.$$

Lemma 4.5 ([17, Chapter III]). *Let G be a lcsc group and let \mathcal{E}, \mathcal{F} be continuous G -modules. Then for any morphism $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ and any two relatively injective strengthened resolutions $0 \longrightarrow \mathcal{E} \longrightarrow (\mathcal{E}_\bullet)$ and $0 \longrightarrow \mathcal{F} \longrightarrow (\mathcal{F}_\bullet)$ there is a lift $\varphi^\bullet: \mathcal{E}_\bullet \rightarrow \mathcal{F}_\bullet$ to a (continuous) cochain morphism, which is unique up to G -equivariant continuous cochain homotopy.*

Definition 4.6 (differential notation, higher order invariants functors). Let G be a lcsc group and \mathcal{E} be a continuous G -module. For each $g \in G$, $\xi \in \mathcal{E}$ we denote $\partial_g: \xi \mapsto g.\xi - \xi$. For $d \in \mathbb{N}$ we define the d 'th order invariants

$$\mathcal{E}^{G(d)} := \{\xi \in \mathcal{E} \mid \forall g_1, \dots, g_d \in G : (\partial_{g_1} \circ \cdots \circ \partial_{g_d}).\xi = 0\}.$$

It is easy to see that $-^{G(d)}$ is a left-exact endo functor on the category of continuous G -modules.

Remark 4.7. Below we will use without reference the observation that $\mathcal{E}^{G(d)}$ is the pre-image in \mathcal{E} of $(\mathcal{E}/\mathcal{E}^{G(d-1)})^G$.

Definition 4.8 (continuous polynomial cohomology). Let G be a lcsc group, let \mathcal{E} be a continuous G -module, and let $d \in \mathbb{N}$. We define the d -th order continuous polynomial cohomology of G with coefficients in \mathcal{E} as

$$H_{(d)}^n(G, \mathcal{E}) := \ker(d^n|_{\mathcal{E}_n^{G(d)}}) / \text{im}(d^{n-1}|_{\mathcal{E}_{n-1}^{G(d)}}), \quad n \in \mathbb{N}_0,$$

where $0 \longrightarrow \mathcal{E} \longrightarrow (\mathcal{E}_\bullet, d^\bullet)$ is any strengthened, relatively injective resolution of \mathcal{E} . By standard arguments [17], using the left-exactness of $-^{G(d)}$ and Lemma 4.5, the polynomial cohomology $H_{(d)}^n(G, \mathcal{E})$ is a vector space with a (not necessarily Hausdorff) vector topology, defined up to (natural, bi-continuous) isomorphism.

Remark 4.9. A direct computation shows that given any topological G -module \mathcal{E} and any $\xi \in \mathcal{E}^{G(2)}$, the map $g \mapsto \partial_g.\xi$ is a continuous homomorphism $G \rightarrow \mathcal{E}$. Thus if G has compact abelianization we conclude that $\mathcal{E}^{G(2)} = \mathcal{E}^G$ for every topological G -module \mathcal{E} , and inductively that $\mathcal{E}^{G(d)} = \mathcal{E}^G$ for all such \mathcal{E} . Hence for such G , the continuous polynomial cohomology coincides with the ordinary continuous cohomology.

More generally, it can be shown that if $d \in \mathbb{N}_0 \cup \{\infty\}$ is the largest extended natural number such that G surjects continuously onto a non-trivial nilpotent group of class d with no compact subgroups, then the continuous polynomial cohomology stabilizes in degree $d+1$.

4.10. Fréchet G -modules. Let G be a lcsc group and let \mathcal{E} be a continuous Fréchet G -module. For $r \in [1, \infty)$ we consider the space $L_{loc}^r(G^n, \mathcal{E})$ of locally r -integrable functions $\xi: G^n \rightarrow \mathcal{E}$. That is, $\xi \in L_{loc}^r(G^n, \mathcal{E})$ if and only if, for any continuous semi-norm q on \mathcal{E} , the composition $\bar{\xi}: G^n \rightarrow \mathcal{E}_q$, where \mathcal{E}_q is the Hausdorff completion of \mathcal{E} wrt. q , is essentially separable-valued and weakly measurable, and $\int_K q(\bar{\xi}(g))^r d\lambda^n(g) < \infty$ for every compact set $K \subseteq G^n$. Then $L_{loc}^r(G^n, \mathcal{E})$, endowed with the projective topology generated by the projections onto $L^r(K, \mathcal{E}_q)$ with $K \subseteq G^n$ ranging over all compact subsets and q over all

continuous semi-norms, is easily seen to be a continuous Fréchet G -module. We will need the following observation, originally due to P. Blanc [7].

Lemma 4.11 ([17, Chapter III]). *Let G be a lcsc group and \mathcal{E} be a continuous Fréchet G -module. Then for every $n \in \mathbb{N}$ and every $r \in [1, \infty)$, $L_{loc}^r(G^n, \mathcal{E})$ is a relatively injective continuous (Fréchet) G -module, and there is a relatively injective, strengthened resolution*

$$0 \longrightarrow \mathcal{E} \longrightarrow L_{loc}^r(G, \mathcal{E}) \longrightarrow \cdots \longrightarrow L_{loc}^r(G^\bullet, \mathcal{E}) \longrightarrow \cdots.$$

It will be convenient later on to consider also G -modules which are not continuous. Let us say that a locally convex space \mathcal{E} is a *locally equicontinuous* G -module if it is endowed with a representation π of G on \mathcal{E} such that $\pi(K) \subset \mathrm{GL}(\mathcal{E})$ is an equicontinuous set for every compact subset K of G .

Definition 4.12 (continuous points). Let G be a lcsc group and let \mathcal{E} be a G -module. We say that a vector $\xi \in \mathcal{E}$ is continuous (wrt. G) if the orbit map $G \ni g \mapsto g.\xi$ is continuous. The set of continuous points in \mathcal{E} is denoted $\mathcal{C}_G\mathcal{E}$, or $\mathcal{C}\mathcal{E}$ when no confusion can arise.

Lemma 4.13. *Let G be a lcsc group and let \mathcal{E} be a locally equicontinuous G -module. Then $\mathcal{C}\mathcal{E}$ is a closed subspace of \mathcal{E} and, with the subspace topology, is a continuous G -module.*

We leave out the straightforward proof. For us the main use of the previous lemma will be the following observation [24]: given any continuous G -module \mathcal{E} , any locally equicontinuous G -module \mathcal{F} , and any G -morphism (that is, continuous linear G -equivariant) $\varphi: \mathcal{E} \rightarrow \mathcal{F}$, the range of φ is contained in $\mathcal{C}\mathcal{F}$.

5. COHOMOLOGY OF NILPOTENT GROUPS

Presently we recall some well-known results concerning the continuous cohomology of nilpotent groups [5, 12, 30]. In this section, the terminology ' \mathcal{H} is a unitary Hilbert G -module' will mean that \mathcal{H} is either a complex Hilbert space with a unitary G -action, or that \mathcal{H} is a real Hilbert space with an orthogonal G -action. All proofs go through verbatim in either case, but we do note that one can always recover the latter case from the former since $H_{(d)}^n(G, \mathcal{E} \otimes \mathbb{C}) \cong H_{(d)}^n(G, \mathcal{E}) \otimes \mathbb{C}$. Thus we will employ without hesitation the above conflation of terminology.

Definition 5.1 (Property H_T [30]). Let G be a lcsc group and $I \subseteq \mathbb{N}$. We say that G has property $H_T(I)$ if for every continuous, unitary Hilbert G -module \mathcal{H} with $\mathcal{H}^G = 0$, and for every $n \in I$, we have

$$\underline{H}^n(G, \mathcal{H}) = 0.$$

We will write H_T instead of $H_T(\mathbb{N})$ when no confusion can arise.

Theorem 5.2 (Delorme [12]). *Let G be a Mal'cev group. For every irreducible, continuous unitary Hilbert G -module \mathcal{H} such that $\mathcal{H}^G = 0$, we have $H^n(G, \mathcal{H}) = 0$ for all $n \in \mathbb{N}_0$. In particular G has property H_T .*

Let us give the proof, essentially as due to Delorme.

Proof. Observe that the latter part of the statement follows from the former: if $\mathcal{H} = \int_{\Omega}^{\oplus} \mathcal{H}_t$ is a direct integral decomposition of an arbitrary (separable) continuous unitary Hilbert G -module \mathcal{H} , it follows by duality arguments (see e.g. [17] for the degree 1 case, the higher

degrees being entirely analogous) that $\underline{H}^n(G, \mathcal{H}) = 0$ if $H^n(G, \mathcal{H}_t) = 0$ for a.e. $t \in \Omega$. Finally, observe that any \mathcal{H} is an increasing union of a net of separable, invariant subspaces.

Let G be a Mal'cev group, let \mathcal{H} be a non-trivial irreducible, continuous unitary Hilbert G -module and denote the G -action by $\pi: G \rightarrow \mathcal{U}(\mathcal{H})$. We prove the claim by induction on $\dim_{\mathbb{R}} \mathfrak{g}$. Let $Z \leq G_{\text{cl}(G)}$ be a closed subgroup, isomorphic to \mathbb{Z} or \mathbb{R} as appropriate, and let $z \in Z$ be the element corresponding to $1 \in \mathbb{Z}$.

Let $\mathcal{A} := \pi(L^1 G)''$ be the von Neumann algebra generated by $\pi(L^1 G)$ in $\mathcal{B}(\mathcal{H})$. Since \mathcal{H} is irreducible, it follows that there are no non-trivial central projections in \mathcal{A} , and so by spectral theory it follows that $\mathcal{Z}(\mathcal{A})$, the center of \mathcal{A} , is trivial, that is $\mathcal{Z}(\mathcal{A}) = \mathbb{C} \cdot 1$.

In particular $\pi|_Z$ acts as multiplication by a character on \mathcal{H} , say, $\pi(z) = \chi(z) \cdot 1$ where $\chi: Z \rightarrow \mathbb{T}$ is a continuous homomorphism. If χ is non-trivial the claim follows directly from [17, Proposition III.3.1]. Hence we may assume that $\chi \equiv 1$, in other words, that $\mathcal{H}^Z = \mathcal{H}$. As G -modules, we then have $H^i(Z, \mathcal{H}) = \mathcal{H}$ for $i = 0, 1$ and vanishing otherwise. In particular, $H^i(Z, \mathcal{H})$ is Hausdorff and so the Hochschild-Serre spectral sequence in continuous cohomology exists and has E_2 -term [17]

$$E_2^{p,q} = H^p(G/Z, H^q(Z, \mathcal{H})) = \begin{cases} H^p(G/Z, \mathcal{H}), & q = 0, 1, \\ 0, & q > 1 \end{cases}.$$

By [17], the cohomology groups are (continuously) subquotients of the (total) E_2 -terms, whence by the induction hypothesis, which gives $H^p(G/Z, \mathcal{H}) = 0$ for all p , we conclude that $H^n(G, \mathcal{H}) = 0$. This finishes the proof. \square

It will be convenient for us to have the following alternate form of Delorme's Theorem.

Theorem 5.3 (Delorme). *Let G be a Mal'cev group and suppose that \mathcal{H} is a continuous, separable unitary Hilbert G -module. Then there is an increasing sequence of closed invariant subspaces $(\mathcal{H}_i)_{i \in \mathbb{N}}$ such that $\cup_i \mathcal{H}_i$ is dense in \mathcal{H} , $\mathcal{H}^G \subseteq \mathcal{H}_i$ for all i , and such that the canonical map induced by inclusion,*

$$H^n(G, \mathcal{H}^G) \rightarrow H^n(G, \mathcal{H}_i)$$

is an isomorphism for all n and all i . (That is, it is continuous, linear, with a continuous linear inverse; In particular, it is part of the conclusion that $H^n(G, \mathcal{H}_i)$ is Hausdorff if \mathcal{H}^G is finite dimensional.)

For later reference, we single out the vanishing property appearing in this version of Delorme's Theorem.

Definition 5.4 (Strong property H_T). Let G be a lcsc group and $I \subseteq \mathbb{N}$. We say that G has strong property H_T in I , denoted $H_T^0(I)$, if for every continuous unitary Hilbert G -module \mathcal{H} such that $\mathcal{H}^G = 0$, there is an increasing net (\mathcal{H}_i) of closed invariant subspaces \mathcal{H}_i of \mathcal{H} , such that $\cup_i \mathcal{H}_i$ is dense in \mathcal{H} , and such that for all $n \in \mathbb{I}$ and all $i \in \mathbb{N}$, we have $H^n(G, \mathcal{H}_i) = 0$.

Remark 5.5. Note that by spectral theory, property H_T^0 is equivalent to property H_T for every countable discrete group Γ with finite classifying space $B\Gamma$.

Proof of Theorem 5.3. Observe first that, taking orthogonal complements, it suffices to prove the statement in the special case $\mathcal{H}^G = 0$. To this end, note that the statement is true for $G = Z$, where Z is either \mathbb{Z} or \mathbb{R} : indeed, in this case we can write \mathcal{H} as a direct integral $\mathcal{H} \cong \int_Z^{\oplus} \mathcal{H}_\chi d\mu$, such that Z acts on each \mathcal{H}_χ by multiplication by the character χ . (Caveat: the measure μ over which we integrate need of course not be the Haar measure on the dual.)

Since in this case $\mathcal{H}^Z = 0$ we have $\mu(\{1_Z\}) = 0$, and by [17, Proposition III.3.1] we may take $\mathcal{H}_i := \int_{Z \setminus U_i}^\oplus$ for any decreasing sequence of relatively compact, open sets U_i such that $\cap_i U_i = \{1_Z\}$ (in \mathbb{T} respectively \mathbb{R}). For a more detailed version of this argument, see Remark 5.6 below.

In general we now proceed by induction on $\dim_{\mathbb{R}} \mathfrak{g}$ as in the proof of Delorme's Theorem given above. Given a Mal'cev group G and a $Z \leq G_{\text{cl}(G)}$ we write $\mathcal{H} = \mathcal{K} \oplus \mathcal{H}^Z$, where \mathcal{K} is the orthogonal complement of \mathcal{H}^Z . Then for $H^n(G, \mathcal{K})$ we disintegrate again \mathcal{K} over Z and take \mathcal{K}_i as in the preceding paragraph, appealing to [17, Proposition III.3.1], and for $H^n(G, \mathcal{H}^Z)$ the claim follows from the induction hypothesis using the Hochschild-Serre spectral sequence in continuous cohomology. \square

Remark 5.6. Let Z be either \mathbb{Z} or \mathbb{R} , $z = 1 \in \mathbb{Z} \leq Z$, and let \mathcal{H} be a unitary Hilbert Z -module, denoting the representation of Z on \mathcal{H} by $\pi: Z \rightarrow \mathcal{U}(\mathcal{H})$. In this remark we give exhaustive details on the computation of $H^\bullet(Z, \mathcal{H})$ and construction of the subspaces \mathcal{H}_i in the first paragraph of the preceding proof. We first explain the case $Z = \mathbb{Z}$. Observe that the cohomology can be computed using the relatively injective resolution

$$0 \longrightarrow \mathcal{H} \xrightarrow{\iota} C(Z, \mathcal{H}) \xrightarrow{d} C(Z, \mathcal{H}) \longrightarrow 0 ,$$

where ι is the embedding as constant functions, and $(d\xi)(n) = \xi(n) - \xi(n-1)$. Taking invariant functions it follows therefore that (as always, $H^0(Z, \mathcal{H}) = \mathcal{H}^Z$) for $n \geq 2$ we have $H^n(Z, \mathcal{H}) = 0$, and that $H^1(Z, \mathcal{H}) \cong \mathcal{H}/(\mathbb{1} - z)\mathcal{H}$. Since $\pi(\mathbb{1} - z) \in \pi(Z)''$ is a normal operator in the von Neumann algebra generated by $\pi(Z)$, the Borel functional calculus implies that, for any compact set $K \subseteq \text{spec}(\pi(\mathbb{1} - z)) \setminus \{0\}$, the operator (where E_K denotes the spectral projection of $\pi(\mathbb{1} - z)$ corresponding to K , that is the operator $\mathbb{1}_K(\pi(\mathbb{1} - z))$ on \mathcal{H} given by the Borel functional calculus) $E_K \cdot \pi(\mathbb{1} - z): E_K \mathcal{H} \rightarrow E_K \mathcal{H}$ is invertible. By [17, Proposition III.3.1] we thus have $H^1(Z, E_K \mathcal{H}) = 0$. Further, since $\mathcal{H}^Z = 0$ we have $\ker \pi(\mathbb{1} - z)\pi(\mathbb{1} - z)^* = \ker \pi(\mathbb{1} - z^{-1}) = 0$ whence $\text{im } \pi(\mathbb{1} - z) = \text{im } \pi(\mathbb{1} - z)\pi(\mathbb{1} - z)^*$ is dense in \mathcal{H} . Thus if we let $K_i \subseteq \text{spec}(\pi(\mathbb{1} - z)) \setminus \{0\}$, $i \in \mathbb{N}$ be compact subsets such that $K_1 \subseteq K_2 \subseteq \dots \subseteq \cup_i K_i = \text{spec}(\pi(\mathbb{1} - z)) \setminus \{0\}$, we have $\cup_i (E_{K_i} \mathcal{H}) = \mathbb{1}_{\text{spec}(\pi(\mathbb{1} - z)) \setminus \{0\}}(\pi(\mathbb{1} - z)) \mathcal{H} = \mathcal{H}$. Note with respect to the second part of the proof of Theorem 5.3 that the subspaces \mathcal{K}_i are by definition equal $E_{K_i} \mathcal{K}$ where $E_{K_i} \in \pi(Z)'' \subseteq \pi(G)''$ are spectral projections of $\pi(\mathbb{1} - z)$ still. In particular the \mathcal{K}_i are thus G -invariant subspaces of \mathcal{K} since $\pi(Z)''$ is contained in the center of $\pi(G)''$.

We observe that the case $Z = \mathbb{Z}$ is in fact sufficient to conclude that proof. Indeed, in the computation of $H^\bullet(G/Z, \mathcal{H}^Z)$ needed for the spectral sequence argument, one can then observe that even in the case $G_{\text{cl}(G)} \cong \mathbb{R}^n$, the center of G/Z now contains a copy of \mathbb{R}/\mathbb{Z} , which can be quotiented out for free since by standard arguments we always have, for any locally compact group H , any compact normal subgroup K , and any Fréchet H -module \mathcal{E} , that $H^\bullet(H, \mathcal{E}) \cong H^\bullet(H/K, \mathcal{E}^K)$. Thus the induction proceeds as before, concluding the proof of 5.3. However, it is instructive in any case to compute $H^\bullet(\mathbb{R}, \mathcal{H})$, so let us give the details for the readers convenience.

We recall from [17, Chapter III.1] that for any Lie group G and any quasi-complete G -module \mathcal{E} , the module of smooth functions $C^\infty(G, \mathcal{E})$ is a relatively injective G -module. In the case of $Z = \mathbb{R}$, there is a relatively injective resolution of \mathcal{H} as follows:

$$0 \longrightarrow \mathcal{H} \xrightarrow{\iota} C^\infty(Z, \mathcal{H}) \xrightarrow{d} C^\infty(Z, \mathcal{H}) \longrightarrow 0 ,$$

where ι is again the embedding as constant functions, and now the coboundary map d is differentiation. Recall also that the representation π of Z in \mathcal{H} induces a derived representation $d\pi$ of the Lie algebra $\mathfrak{z} = \mathbb{R}$ of Z , on the space of smooth vectors \mathcal{H}^∞ in \mathcal{H} . For some details and further references on this we refer to [17]. The point is that there is an unbounded, densely defined self-adjoint operator $d\pi(y)$, where $\mathfrak{z} \ni y = \log(z)$, acting on \mathcal{H} , affiliated with $\pi(Z)''$, and such that $\pi(z) = \exp(d\pi(y))$. Further, since Z is one-dimensional, the space \mathcal{H}^∞ of smooth vectors in \mathcal{H} is precisely the domain of $d\pi(y)$. Given all this, applying the invariants functor to the relatively injective resolution above, it is not hard to see that $H^1(Z, \mathcal{H}) \cong \mathcal{H}^\infty / \text{im}(d\pi(y))$. By essentially the same argument as above, we then get the result by taking spectral projections, say $P_n := \mathbb{1}_{(-\infty, 1/n] \cup [1/n, +\infty)}(d\pi(y))$, $n \in \mathbb{N}$, for which again $P_n \nearrow \mathbb{1}$ and $H^1(Z, P_n \cdot \mathcal{H}) = 0$.

The following corollary provides a very direct and useful extension of Delorme's Theorem. The important case, as is clear after the exposition in the following two sections, is the case where $\mathcal{F} = C(G, \mathbb{R})^{G(d)}$ for some $d \in \mathbb{N}$, as this corresponds by the results in Section 6 to the polynomial cohomology with coefficients in \mathcal{H} .

Corollary 5.7. *Let G be a lcsc group with property H_T^0 , e.g. a Mal'cev group, and $\mathcal{E} = \mathcal{H} \otimes \mathcal{F}$ with \mathcal{H} a continuous unitary Hilbert G -module, and \mathcal{F} a continuous G -module such that $\mathcal{F} = \mathcal{F}^{G(d)}$ for some $d \in \mathbb{N}$ and $\dim \mathcal{F} < \infty$. Then the natural inclusion map $\mathcal{H}^G \otimes \mathcal{F} \rightarrow \mathcal{E}$ induces an isomorphism*

$$H^n(G, \mathcal{H}^G \otimes \mathcal{F}) \xrightarrow{\cong} \underline{H}^n(G, \mathcal{E}).$$

Proof. Indeed, denoting $\mathcal{K} := (\mathcal{H}^G)^\perp$ we have

$$H^n(G, \mathcal{E}) = H^n(G, \mathcal{K} \otimes \mathcal{F}) \oplus H^n(G, \mathcal{H}^G \otimes \mathcal{F}).$$

Choosing a sequence $(K_i)_i$ of invariant subspaces of \mathcal{K} as in the theorem, we have by induction on d and the long exact sequence in cohomology that $H^n(G, \mathcal{K}_i \otimes \mathcal{F}) = 0$ for all i . Since $\mathcal{K} \otimes \mathcal{F}$ admits a continuous G -equivariant projection $P_{\mathcal{K}_i} \otimes \mathbb{1}$ onto $\mathcal{K}_i \otimes \mathcal{F}$ for all i , it follows that $\underline{H}^n(G, \mathcal{K}_i \otimes \mathcal{F}) = 0$. \square

As a small digression, let us observe that the vanishing of reduced cohomology in Delorme's Theorem can be extended quite a bit, thanks to [5]:

Theorem 5.8. *Let G be a Mal'cev group. For any weakly almost periodic (continuous) isometric Banach G -module \mathcal{E} such that $\mathcal{E}^G = 0$ we have*

$$\underline{H}^n(G, \mathcal{E}) = 0, \quad n \in \mathbb{N}.$$

Definition 5.9. In general, we will say that a lcsc group such that $\underline{H}^n(G, \mathcal{E}) = 0$ for every weakly almost periodic, continuous, isometric Banach G -module \mathcal{E} and all $n \in I$ has property $H_T^{w\text{ap}}(I)$.

Proof of Theorem 5.8. We proceed by induction on $\dim \mathfrak{g}$, the case $\dim \mathfrak{g} = 1$ being clear (e.g. by [5]). Given a Mal'cev group G with $\dim \mathfrak{g} > 1$, let $Z \leq G_{\text{cl}(G)}$ be a closed subgroup, isomorphic to either \mathbb{Z} or \mathbb{R} as appropriate. Let \mathcal{E} be as in the statement above. By the Alaoglu-Birkhoff Decomposition Theorem [1] there is a G -equivariant decomposition $\mathcal{E} \cong \mathcal{F} \oplus_1 \mathcal{E}^Z$ with $\mathcal{F} \leq \mathcal{E}$ consisting of the set of vectors $\xi \in \mathcal{E}$ such that $0 \in \overline{\text{conv}}(Z \cdot \xi)$. In particular $\mathcal{F}^Z = 0$. Then we decompose (bi-continuously) as well the cohomology $H^n(G, \mathcal{E}) = H^n(G, \mathcal{F}) \oplus H^n(G, \mathcal{E}^Z)$ and note that by [5, Theorem 2] we have $\underline{H}^n(G, \mathcal{F}) = 0$.

To handle the second term, we observe that, as G -modules, we have $H^i(Z, \mathcal{E}^Z) = \mathcal{E}^Z$ for $i = 0, 1$ and vanishing otherwise. In particular, $H^i(Z, \mathcal{E}^Z)$ is Hausdorff and so the Hochschild-Serre spectral sequence in continuous exists and has E_2 -term [17]

$$E_2^{p,q} = H^p(G/Z, H^q(Z, \mathcal{E}^Z)) = \begin{cases} H^p(G/Z, \mathcal{E}^Z), & q = 0, 1, \\ 0, & q > 1 \end{cases}.$$

By [17], the cohomology groups are (continuously) subquotients of the (total) E_2 -terms, whence by the induction hypothesis, which gives $\underline{H}^p(G/Z, \mathcal{E}^Z) = 0$ for all p , we conclude that $\underline{H}^n(G, \mathcal{E}^Z) = 0$. This finishes the proof. \square

In the sequel we will need a further generalization of Delorme's result. To wit, we will consider in particular certain continuous G -modules \mathcal{E} , for which $\mathcal{E}^G \neq \mathcal{E}^{G(2)}$, but where \mathcal{E} does not seem to admit a decomposition as in Corollary 5.7. But since $\mathcal{E}^{G(2)}$ is the pre-image in \mathcal{E} of $(\mathcal{E}/\mathcal{E}^G)^G$ it follows by [24, Lemma 1.2.10] that \mathcal{E} cannot possibly be isometric. To state the result it will be convenient to formally single out the modules we will consider.

Definition 5.10 (poly-Hilbert module). Let G be a lcsc group. We say that a continuous Fréchet G -module \mathcal{E} is a *poly-Hilbert* module (respectively *poly-wap* module, respectively *poly-isometric*) if there is a sequence of closed G -invariant subspaces $0 = \mathcal{E}_0 \leq \mathcal{E}_1 \leq \mathcal{E}_2 \leq \dots \leq \mathcal{E}_d = \mathcal{E}$, such that for every i , $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a unitary Hilbert G -module (respectively the subquotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are Banach spaces and the G -action is isometric and weakly almost periodic, respectively the subquotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are isometric Fréchet G -modules).

Such a sequence (\mathcal{E}_i) is called a composition series of \mathcal{E} and d its degree. Formally, when we say 'let \mathcal{H} be a poly-Hilbert G -module', part of the data is a given composition series, that is the data is really the pair $(\mathcal{H}, (\mathcal{H}_i))$.

The naive idea when trying to compute the cohomology with coefficients in, say, a poly-Hilbert G -module \mathcal{H} , is to proceed inductively, using the long exact sequence in continuous cohomology and thus reduce to a series of computations of cohomology with coefficients in Hilbert G -modules and certain maps between these. However, typically we might want to conclude vanishing of the *reduced* cohomology with coefficients in \mathcal{H} , but the long exact sequence in cohomology does not in general induce an *exact* sequence in the reduced cohomology spaces. In order to get around this problem we will need the following construction.

Let G be a lcsc group with property H_T^0 , and let \mathcal{H} be a continuous, separable poly-Hilbert G module with composition series (\mathcal{H}_i) . For each $i = 1, \dots, d$ let $(\mathcal{K}_j^{(i)})_j$ be a fixed sequences of closed, G -invariant subspaces of $\mathcal{H}_i/\mathcal{H}_{i-1}$ as in Theorem 5.3, that is, $(\mathcal{K}_j^{(i)})_j$ is an increasing sequence with union dense in $\mathcal{H}_i/\mathcal{H}_{i-1}$ and such that $(\mathcal{H}_i/\mathcal{H}_{i-1})^G \subseteq K_j^{(i)}$ for all j . For stupid technical reasons we will need the following construction. We claim that there is a separable, continuous Fréchet G -module \mathcal{F} such that \mathcal{H} embeds continuously in \mathcal{F} with dense image, and such that, letting $\mathcal{F}_i = \overline{\iota(\mathcal{H}_i)}$, we have for all $i = 1, \dots, d$

$$\mathcal{F}_i/\mathcal{F}_{i-1} = \lim_{\leftarrow j} \mathcal{K}_j^{(i)}.$$

We construct \mathcal{F} inductively. The construction is clear for $d = 1$. Having constructed \mathcal{F}_{d-1} satisfying the above for \mathcal{H}_{d-1} in place of \mathcal{H} , we construct \mathcal{F} as follows. Denote the canonical projection $\mathcal{H} \rightarrow \bar{\mathcal{H}} := \mathcal{H}/\mathcal{H}_{d-1}$ by π and let $\sigma: \bar{\mathcal{H}} \rightarrow \mathcal{H}_{d-1}$ be a continuous section as given by the Michael Selection Theorem and denote $\kappa: \mathcal{H} \rightarrow \mathcal{H}: \xi \mapsto \xi - (\sigma \circ \pi)(\xi)$. Consider the

maps

$$\begin{aligned}\varphi: \mathcal{H} &\rightarrow \bar{\mathcal{H}} \times \mathcal{H}_{d-1}: \xi \mapsto (\pi(\xi), \kappa(\xi)), \\ \varphi': \bar{\mathcal{H}} \times \mathcal{H}_{d-1} &\rightarrow \mathcal{H}: (\bar{\xi}, \eta) \mapsto \sigma(\bar{\xi}) + \eta.\end{aligned}$$

A direct computation shows that $\varphi' = \varphi^{-1}$, and so we get a homeomorphism $\mathcal{H} \cong \bar{\mathcal{H}} \times \mathcal{H}_{d-1}$. On the right-hand side, the G -action can be expressed as

$$g \cdot (\bar{\xi}, \eta) := \varphi(g \cdot \varphi'(\bar{\xi}, \eta)) = (g \cdot \bar{\xi}, \alpha(g, \bar{\xi}) + g \cdot \eta),$$

where $\alpha: G \times \bar{\mathcal{H}} \rightarrow \mathcal{H}_{d-1}$ is the continuous cocycle given by $\alpha(g, \bar{\xi}) = \kappa(g \cdot \sigma(\bar{\xi}))$. In this way, we can extend the G -action continuously to the intermediate space $\mathcal{F}' := \bar{\mathcal{H}} \times \mathcal{F}_{d-1}$. Similarly, the addition and scalar multiplication on \mathcal{H} extend continuously to \mathcal{F}' so that this is a continuous Fréchet G -module. Now denote by $\pi: \mathcal{F}' \rightarrow \bar{\mathcal{H}}$ the canonical projection, and take

$$\mathcal{F} := \varprojlim_j \pi^{-1}(\mathcal{K}_j^{(d)}).$$

Note that, if we write $\mathcal{L}_1^{(d)} := \mathcal{K}_1^{(d)}$ and inductively $\mathcal{L}_j^{(d)} := \left(\oplus_{k=1}^{j-1} \mathcal{L}_k^{(d)}\right)^\perp \cap \mathcal{K}_j^{(d)}$, the definition gives that $\mathcal{F}_d/\mathcal{F}_{d-1} = \prod_j \mathcal{L}_j^{(d)}$. In particular, we observe that (by induction) we have for all $i = 1, \dots, d$ that

$$(5.1) \quad H^n(G, \mathcal{F}_i/\mathcal{F}_{i-1}) = H^n(G, (\mathcal{H}_i/\mathcal{H}_{i-1})^G).$$

In particular, if $(\mathcal{H}_i/\mathcal{H}_{i-1})^G = 0$ for all i it follows that $H^n(G, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ – that is we get vanishing of cohomology on the nose, not just of reduced cohomology.

Remark 5.11. Alternatively, given \mathcal{F}_{d-1} we can construct \mathcal{F} as follows. For simplicity we denote $\mathcal{K} := \mathcal{H}_{d-1}$, considered as a continuous G -module with the subspace topology induced by the inclusion $\mathcal{K} \subseteq \mathcal{F}_{d-1}$. By definition \mathcal{F}_{d-1} is the completion of \mathcal{K} .

Now observe that the topology on \mathcal{K} can be induced by a continuous metric on \mathcal{H} . Indeed, it is easy to see that we can find a (countable) neighborhood basis $\{U_i\}_{i \in \mathbb{N}}$ of zero in \mathcal{K} such that $U_i = \mathcal{K} \cap U'_i$ where each U'_i is a convex neighborhood of zero in \mathcal{H} .

Similarly we consider the morphism $\pi: \mathcal{H} \rightarrow \prod_j \mathcal{L}_j^{(d)}$ where $\bar{\mathcal{H}} = \oplus_j \mathcal{L}_j^{(d)}$ as above. Letting $\{V'_i\}_{i \in \mathbb{N}}$ be a convex neighborhood basis of zero in $\prod_j \mathcal{L}_j^{(d)}$ we define $V_i := \pi^{-1}(V'_i)$, $i \in \mathbb{N}$. Now consider on \mathcal{H} the topology generated by $\{U_i\}_i \cup \{V_i\}_i$ and denote the resulting continuous G -module by \mathcal{F}' . It is clear that the completion of \mathcal{F}' is precisely \mathcal{F} as needed.

Definition 5.12 (H_T^0 -completion). Given G and \mathcal{H} as above, we call the continuous G -module \mathcal{F} constructed above the H_T^0 -completion of $(\mathcal{H}, (\mathcal{K}_j^{(i)}))$. Below we will abuse terminology whenever possible and write simply 'let \mathcal{F} be the/an H_T^0 -completion of \mathcal{H} ' when no confusion can arise.

Note that the topology on the H_T^0 -completion \mathcal{F} of \mathcal{H} as in the previous definition does not depend on the choice of section(s) in the construction.

Definition 5.13. Let G be a lcsc group. We say that G is *cohomologically finite dimensional* if $\dim_{\mathbb{R}} H^n(G, \mathcal{E}) < \infty$ for every continuous finite dimensional continuous G -module \mathcal{E} and every $n \in \mathbb{N}$.

Remark 5.14. (i) Every countable group Γ with finite classifying space $B\Gamma$ is cohomologically finite dimensional.

- (ii) Every connected Lie group G is cohomologically finite dimensional. This follows from the van Est Theorem [33, 34].
- (iii) Mal'cev groups are cohomologically finite dimensional. Indeed let G be a Mal'cev group and let \mathcal{E} be a continuous finite dimensional G -module. It is easy to see that $H^n(G, \mathcal{E})$ is finite dimensional for all $n \in \mathbb{N}$, using the Hochschild-Serre spectral sequence in continuous cohomology. Alternatively, using the smooth version of Shapiro's Lemma in [17, part III], one can reduce to Lie algebra cohomology and apply the spectral sequence there.

Theorem 5.15. *Let G cohomologically finite dimensional, lcsc group with property H_T^0 . Suppose that \mathcal{H} is a continuous poly-Hilbert G -module such that $\dim_{\mathbb{R}} \mathcal{H}^{G(d)} < \infty$, and with a composition series $(\mathcal{H}_i)_{i=1, \dots, d}$ such that:*

- (i) *For every $i = 1, \dots, d$ we have $\mathcal{H}^{G(i)} \leq \mathcal{H}_i$, and the natural map $\mathcal{H}^{G(i)} / \mathcal{H}^{G(i-1)} \rightarrow (\mathcal{H}_i / \mathcal{H}_{i-1})^G$ is an isomorphism.*

Let \mathcal{F} be any H_T^0 -completion of \mathcal{H} . Then in the commutative diagram

$$\begin{array}{ccc} H^n(G, \mathcal{H}^{G(d)}) & \xrightarrow{\quad} & \underline{H}^n(G, \mathcal{H}) , \\ & \searrow \iota_* \quad \swarrow & \\ & H^n(G, \mathcal{F}) & \end{array}$$

the natural map $\iota_: H^n(G, \mathcal{H}^{G(d)}) \rightarrow H^n(G, \mathcal{F})$ is an isomorphism. Moreover, Suppose that instead of (i), the pair \mathcal{H}, \mathcal{F} satisfies the (a priori weaker) assumption as follows:*

- (ii) *We have $\dim_{\mathbb{C}} \mathcal{F}^{G(d)} < +\infty$, and for every $i = 1, \dots, d$ we have $\mathcal{F}^{G(i)} \leq \mathcal{F}_i$, and the natural map $\mathcal{F}^{G(i)} / \mathcal{F}^{G(i-1)} \rightarrow (\mathcal{F}_i / \mathcal{F}_{i-1})^G$ is an isomorphism.*

Then the natural map $\iota_: H^n(G, \mathcal{F}^{G(d)}) \rightarrow H^n(G, \mathcal{F})$ is an isomorphism.*

Proof. To prove the theorem we proceed by induction on d , noting that the case $d = 1$ follows directly from the Delorme Theorem 5.2 and the construction of \mathcal{F} . Fix a $d > 1$. Observe that, by construction we have $(\mathcal{F}_i / \mathcal{F}_{i-1})^G = (\mathcal{H}_i / \mathcal{H}_{i-1})^G$. Note that \mathcal{F} satisfies the hypotheses of the theorem, with 'poly-isometric' replacing 'poly-Hilbert'. Then so does $\mathcal{E} := \mathcal{F} / \mathcal{F}^{G(d)}$, with $\mathcal{E}_i := \bar{\mathcal{F}}_i$, the image of \mathcal{F}_i under the canonical projection: by the construction of \mathcal{F} and the hypothesis (i), it follows that $(\mathcal{E}_i / \mathcal{E}_{i-1})^G = 0$ for all $i = 1, \dots, d$: indeed, suppose that $\xi \in \mathcal{F}_i$ such that $\bar{\xi} \in (\mathcal{E}_i / \mathcal{E}_{i-1})^G$; by definition, this means that $\partial_g . \xi \in \mathcal{F}_{i-1} + \mathcal{F}^{G(i)}$ for all g , that is, if $\pi: \mathcal{F}_i \rightarrow \mathcal{F}_i / \mathcal{F}_{i-1}$ denotes the canonical projection, that $\partial_g . \pi(\xi) \in \pi(\mathcal{F}^{G(i)}) = (\mathcal{F}_i / \mathcal{F}_{i-1})^G$ for all g , where the equality follows from hypothesis (i) in the statement. By construction, $(\mathcal{F}_i / \mathcal{F}_{i-1})^{G(2)} = (\mathcal{F}_i / \mathcal{F}_{i-1})^G$, since the quotient is a product $\mathcal{F}_i / \mathcal{F}_{i-1} = \prod_j \mathcal{L}_j$ of G -modules \mathcal{L}_j for which $\mathcal{L}_j^{G(2)} = \mathcal{L}_j^G$ by [24, Lemma 1.2.10]. Thus $\pi(\xi) \in (\mathcal{F}_i / \mathcal{F}_{i-1})^G$, that is, $\bar{\xi} = 0$ in $\mathcal{E}_i / \mathcal{E}_{i-1}$.

We claim first that $H^n(G, \mathcal{E}) = 0$ for all n . To see this, note that \mathcal{E} is the H_T^0 -completion of $\mathcal{H} / \mathcal{H}^{G(d)}$. The claim then follows from (5.1) and induction on d , using the long exact sequence in continuous cohomology induced by $0 \longrightarrow \mathcal{E}_{d-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} / \mathcal{E}_{d-1} \longrightarrow 0$.

Considering the long exact sequence in continuous cohomology induced by the short exact sequence

$$0 \longrightarrow \mathcal{H}^{G(d)} \xrightarrow{\iota} \mathcal{F} \xrightarrow{\pi} \mathcal{E} \longrightarrow 0 ,$$

we get in degree n an exact sequence

$$H^{n-1}(G, \mathcal{E}) \xrightarrow{\delta} H^n(G, \mathcal{H}^{G(d)}) \xrightarrow{\iota_*} H^n(G, \mathcal{F}) \xrightarrow{\pi_*} H^n(G, \mathcal{E}).$$

As we just observed above, $H^{n-1}(G, \mathcal{E}) = H^n(G, \mathcal{E}) = 0$ and so ι_* is a continuous linear bijection. Further, since \mathcal{F} is Fréchet and $H^n(G, \mathcal{F})$ is finite dimensional (since ι_* is surjective and G is cohomologically finite dimensional), it follows [17, Lemme D.1(ii)] that $H^n(G, \mathcal{F})$ is Hausdorff. By uniqueness of the vector space topology on a finite dimensional vector space, i_* is then an isomorphism. This finishes the proof in case (i), and case (ii) follows by a similar argument. \square

Remark 5.16. We do not know if in fact the map $H^n(G, \mathcal{H}^{G(d)}) \rightarrow \underline{H}^n(G, \mathcal{H})$ is always an isomorphism.

6. COHOMOLOGICAL INDUCTION THROUGH A MEASURE EQUIVALENCE

In this section we prove a "reciprocity theorem", analogous to [25, Proposition 4.6], in the context of polynomial cohomology. As an application of this, using the self-coupling (G, λ) of a lcscu group G , we derive a description of the polynomial cohomology $H_{(d)}^\bullet(G, \mathcal{E})$ in terms of linear cohomology.

6.1. Function spaces on Ω . Let (Ω, μ) be a standard Borel measure space. In particular Ω is σ -finite, and furthermore we can find an increasing sequence $\Omega_k \subseteq \Omega$ of relatively compact subsets such that for any compact set $K \subseteq \Omega$ we have $\mu(K \setminus \Omega_k) = 0$ for all sufficiently large k . Thus for any $1 \leq r < \infty$ and any Fréchet space \mathcal{E} , we can unambiguously speak of the space $L_{loc}^r(\Omega, \mathcal{E})$. Recall that this is the projective limit $\lim_{\leftarrow} L^r(\Omega_k, \mathcal{E}_q)$ over $k \in \mathbb{N}$ and q running through a net of continuous semi-norms on \mathcal{E} , separating points, and thus generating the topology. Note that this is a complete G -module, and is a Fréchet space if \mathcal{E} is. Analogously we define $L_{loc}^\infty(\Omega, \mathcal{E})$.

Suppose now that (Ω, μ) is a (G, H) -coupling for lcscu groups G, H . For any continuous Fréchet $G \times H$ -module \mathcal{E} , consider $L_{loc}^r(\Omega, \mathcal{E})$ with the canonical $G \times H$ -module structure given by

$$((g, h).f)(t) = (g, h).f((g, h)^{-1}.t), \quad f \in L_{loc}^r(\Omega, \mathcal{E}), t \in \Omega.$$

More generally, for any $n \in \mathbb{N}_0$ we can consider the $G \times H$ -module $L_{loc}^r(G^n \times \Omega, \mathcal{E})$ with the action

$$((g, h).f)(g_1, \dots, g_n, t) = (g, h).f(g^{-1}g_1, \dots, g^{-1}g_n, (g, h)^{-1}.t), \quad f \in L_{loc}^r(\Omega, \mathcal{E}), g_i \in G, t \in \Omega$$

and analogously for $L_{loc}^r(H^n \times \Omega, \mathcal{E})$.

Lemma 6.2. *For any $n \in \mathbb{N}_0$, $1 \leq r < \infty$, and any continuous Fréchet $G \times H$ -module \mathcal{E} , $L_{loc}^r(G^{n+1} \times \Omega, \mathcal{E})$ is a continuous, relatively injective Fréchet $G \times H$ -module. So is $L_{loc}^r(H^{n+1} \times \Omega, \mathcal{E})$.*

Proof. Indeed, since j, j^{-1} are bounded and equivariant, we have

$$L_{loc}^r(G^{n+1} \times \Omega, \mathcal{E}) \cong L_{loc}^r(G \times H, L_{loc}^r(G^n \times \Omega, \mathcal{E})),$$

and similarly for H . The lemma then follows by [17, Proposition III.1.4]. \square

6.3. A reciprocity theorem. As a consequence of the previous lemma, we can compute the polynomial cohomology of $G \times H$ with coefficients in $L_{loc}^r(\Omega, \mathcal{E})$ using either the strengthened, relatively injective resolution

$$(6.1) \quad 0 \longrightarrow L_{loc}^r(\Omega, \mathcal{E}) \xrightarrow{\iota} L_{loc}^r(G, L_{loc}^r(\Omega, \mathcal{E})) \xrightarrow{d_G^0} L_{loc}^r(G^2, L_{loc}^r(\Omega, \mathcal{E})) \xrightarrow{d_G^1} \dots,$$

or the strengthened, relatively injective resolution

$$(6.2) \quad 0 \longrightarrow L_{loc}^r(\Omega, \mathcal{E}) \xrightarrow{\iota} L_{loc}^r(H, L_{loc}^r(\Omega, \mathcal{E})) \xrightarrow{d_H^0} L_{loc}^r(H^2, L_{loc}^r(\Omega, \mathcal{E})) \xrightarrow{d_H^1} \dots,$$

where in both cases, the coboundary maps are the usual ones from the bar resolutions of G , respectively H -modules. The following result is then the direct analogue of [25, Theorem 4.6]

Theorem 6.4 (reciprocity). *Let $d, d' \in \mathbb{N}$, let G, H be lcscu groups, let (Ω, μ) be a (G, H) -coupling, $1 \leq r < \infty$, and let \mathcal{E} be a continuous Fréchet $G \times H$ -module. Then there is a chain map χ^\bullet from (6.1) to (6.2), lifting the identity map on $L_{loc}^r(\Omega, \mathcal{E})$, given by (where χ_G is as in (3.2))*

$$(\chi^n f)(h_0, \dots, h_n)(t) = f(\chi_G(h_0^{-1}t), \dots, \chi_G(h_n^{-1}t))(t).$$

Further, χ^\bullet descends to an isomorphism

$$\chi^\bullet: H_{(d)}^\bullet(G, L_{loc}^r(\Omega, \mathcal{E})^{H(d')}) \xrightarrow{\cong} H_{(d')}^\bullet(H, L_{loc}^r(\Omega, \mathcal{E})^{G(d)})$$

Proof. As we noted when we defined $\chi_G: \Omega \rightarrow G$ in Section 3, it is a locally bounded Borel map, and it follows from this that the maps $\chi^n: L_{loc}^r(G^{n+1}, L_{loc}^r(\Omega, \mathcal{E})) \rightarrow L_{loc}^r(H^{n+1}, L_{loc}^r(\Omega, \mathcal{E}))$ are well-defined, continuous $G \times H$ -morphisms; clearly they commute with the coboundary maps as well. Now consider the left-exact "higher order bi-invariants" functor $G(d) \times H(d')$ on the category of continuous $G \times H$ -modules, defined by

$$\mathcal{E}^{G(d) \times H(d')} = \mathcal{E}^{G(d)} \cap \mathcal{E}^{H(d')}.$$

We denote the derived functors by $H_{(d,d')}^\bullet(G \times H, -)$. By the above, we can compute these in $L_{loc}^r(\Omega, \mathcal{E})$ using the resolutions (6.1), respectively (6.2), and it remains to identify

$$H_{(d)}^\bullet(G, L_{loc}^r(\Omega, \mathcal{E})^{H(d')}) \cong H_{(d,d')}^\bullet(G \times H, L_{loc}^r(\Omega, \mathcal{E})) \cong H_{(d')}^\bullet(H, L_{loc}^r(\Omega, \mathcal{E})^{G(d)}),$$

using the complexes (6.1), respectively (6.2), which is clear. \square

Remark 6.5. Note that in general $\mathcal{E}^{G(d)} \cap \mathcal{E}^{H(d')}$ need not equal $\mathcal{E}^{(G \times H)(d)}$.

6.6. The tautological self-coupling. Recall that any lcscu group G induces a (uniform) (G, G) -coupling, called the tautological (self-)coupling, by setting $(\Omega, \mu) = (G, \lambda)$ and letting $G \times G$ act by $(g, h).t = gth^{-1}$. Presently we note the following modification of the reciprocity theorem just proved to give a description of polynomial cohomology with trivial coefficients in terms of linear cohomology. In the statement we will write $\text{Pol}_{d-1}(G)$ for the space $C(G, \mathbb{R})^{G(d)}$ with respect to the right-regular representation. See the introduction, or Section 7 below for explanations of this terminology.

Observation 6.7. Let G be a lcsc group and $\xi: G \rightarrow \mathbb{R}$ be a Borel map such that $\xi \in \{\xi: G \rightarrow \mathbb{R}\}^{G(d)}$ for some $d \in \mathbb{N}$. We claim that ξ is in fact continuous; to prove this we proceed by induction on d . For $d = 1$ this is trivial since ξ is then constant. For $d = 2$, ξ is an affine homomorphism, that is, a homomorphism plus a constant translation. In this case

the claim follows from a well-known argument of Banach. In fact, recall that if $\eta: G \rightarrow \mathcal{E}$ is a Borel 1-cocycle into a continuous, separable Fréchet G -module \mathcal{E} , then ξ is continuous. (We will apply a very similar argument below, and thus will not recall Banach's argument here.)

Now we consider the map $\alpha_\xi: g \mapsto \mathbb{G}_g \xi$ taking values, by the induction hypothesis, in $C(G, \mathbb{R})$. Further, a direct computation shows that, considering $C(G, \mathbb{R})$ with the right-regular representation, α_ξ satisfies the 1-cocycle identity: $\alpha_\xi(gh) = \rho(g) \cdot \alpha_\xi(h) + \alpha_\xi(g)$ for all $g, h \in G$. Another direct computation shows that α_ξ is weakly Borel, that is, for each $\varphi \in C(G, \mathbb{R})^* \cong \lim_{\rightarrow} C(K, \mathbb{R})^*$ the function $g \mapsto \varphi(\alpha_\xi(g))$ is Borel. Since $C(G, \mathbb{R})$ is a separable Fréchet space the weak and strong Borel structures coincide, and so α_ξ is strongly Borel. By the argument of Banach, it follows that α_ξ is continuous.

Now let $K \subseteq G$ be a compact set, not of measure zero, and such that $\xi|_K$ is continuous. The existence of such a set is guaranteed by Lusin's Theorem. We want to prove that for sequences $(x_n), (y_n) \subseteq K$ converging to x respectively y in K , we have $\xi(x_n y_n) \rightarrow_n \xi(xy)$. By a "subsequences of subsequences" argument, using the compactness of K , this is easily seen to imply that ξ is continuous on K^2 . To see the claim, we write

$$(6.3) \quad \xi(x_n y_n) = (\mathbb{G}_{y_n} \xi)(x_n) + \xi(x_n).$$

By the continuity of α_ξ (and compactness of K) it follows that $(\mathbb{G}_{y_n} \xi)(x_n) \rightarrow_n (\mathbb{G}_y \xi)(x)$, and so we conclude that indeed $\xi(x_n y_n) \rightarrow_n \xi(xy)$. By a well-known result of Weil, K^2 contains an open set U . Using again (6.3) it follows that ξ is continuous on all of G .

Proposition 6.8. *Let G be a lcsc group, $d \in \mathbb{N}$. Then:*

(i) *There are isomorphisms $\tau^\bullet: H_{(1)}^\bullet(G, \text{Pol}_{d-1}(G)) \xrightarrow{\cong} H_{(d)}^\bullet(G, \mathbb{R})$, given on cochains by*

$$(\tau^n \xi)(g_0, \dots, g_n) = \xi(g_0, \dots, g_n)(\mathbb{1})$$

and with inverse

$$(\tau^n)^{-1}(\xi)(g_0, \dots, g_n)(t) = \xi(t^{-1}g_0, \dots, t^{-1}g_n).$$

(ii) *More generally, let $G_1 = G_2 = G$ and suppose, for simplicity, further that G is unimodular. For any coefficient G -module $G \curvearrowright^\pi \mathcal{E}$, endowing this with the $G_1 \times G_2$ -module structure given by $(g_1, g_2) \cdot \xi = \pi(g_1) \cdot \xi$, the map χ from the Reciprocity Theorem 6.4 gives an isomorphism*

$$\chi^\bullet: H^\bullet(G_1, L_{loc}^1(G, \mathcal{E})^{G_2(d)}) \xrightarrow{\cong} H_{(d)}^\bullet(G_2, L_{loc}^1(G, \mathcal{E})^{G_1}) = H_{(d)}^\bullet(G, \mathcal{E}).$$

Furthermore, note that $H^\bullet(G_1, L_{loc}^1(G, \mathcal{E})^{G_2(d)}) = H^\bullet(G, \text{Pol}_{d-1}(G) \hat{\otimes} \mathcal{E})$ where $\text{Pol}_{d-1}(G) \hat{\otimes} \mathcal{E}$ denotes the projective tensor product of $\text{Pol}_{d-1}(G) \leq L_{loc}^1(G, \mathbb{R})$. In particular, if G is residually Mal'cev, then $\dim_{\mathbb{R}} \text{Pol}_d(G)$ is finite for every d , and so we can write simply $H^\bullet(G_1, L_{loc}^1(G, \mathcal{E})^{G_2(d)}) = H^\bullet(G, \text{Pol}_{d-1}(G) \otimes \mathcal{E})$

Proof. Part (i) follows by exactly the same argument as in the proof of Theorem 6.4, replacing everywhere L_{loc}^r -spaces by spaces of continuous functions. Part (ii) follows directly by the Reciprocity Theorem 6.4. \square

Remark 6.9. Let us denote $\tau' := (\tau^1)^{-1}$. Using notation from Section 7 below (see Proposition 7.1), we can describe τ' explicitly on the level of inhomogeneous cocycles. We have for

a continuous homogeneous polynomial 1-cocycle ξ that, for $g, h \in G$,

$$\begin{aligned} \overline{(\tau'\xi)}(g)(h) &= (\tau'\xi)(1, g)(h) \\ &= \xi(h^{-1}, h^{-1}g) \\ &= \bar{\xi}(h^{-1}g) - \bar{\xi}(h^{-1}) \\ &= (\mathbb{G}_g \bar{\xi})(h^{-1}) = \left(\partial_g \bar{\xi} \right) (h). \end{aligned}$$

Corollary 6.10. *Let G be a cohomologically finite dimensional lcsc group. Then $\dim_{\mathbb{R}} \text{Pol}_d(G) < +\infty$ for all $d \in \mathbb{N}$.*

Proof. By induction on d . Observe first that $\dim_{\mathbb{R}} \text{Pol}_1(G) = 1 + \dim_{\mathbb{R}} H^1(G, \mathbb{R}) < +\infty$. The inductive step follows then directly from part (i) of the proposition. \square

Corollary 6.11 (long exact sequence in polynomial cohomology). *Let G be a lcsc(u) group and let $0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{F} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0$ be a short exact sequence of continuous Fréchet G -modules. Then for every $d \in \mathbb{N}$ there is a long exact sequence in polynomial cohomology*

$$\cdots \xrightarrow{\delta} H_{(d)}^{\bullet}(G, \mathcal{E}) \xrightarrow{\iota_*} H_{(d)}^{\bullet}(G, \mathcal{F}) \xrightarrow{\pi_*} H_{(d)}^{\bullet}(G, \mathcal{Q}) \xrightarrow{\delta} H_{(d)}^{\bullet+1}(G, \mathcal{E}) \xrightarrow{\iota_*} \cdots,$$

where all maps ι_*, π_*, δ are continuous.

Proof. Given a short exact sequence of continuous Fréchet G -modules as in the statement, it is easy to see that the corresponding sequence

$$0 \longrightarrow \text{Pol}_d(G) \hat{\otimes} \mathcal{E} \xrightarrow{\iota} \text{Pol}_d(G) \hat{\otimes} \mathcal{F} \xrightarrow{\pi} \text{Pol}_d(G) \hat{\otimes} \mathcal{Q} \longrightarrow 0,$$

where $\text{Pol}_d(G) \leq L_{loc}^1(G)$, is exact as well. Hence the statement follows directly from the long exact sequence in continuous cohomology [17] and Proposition 6.8(ii). \square

Remark 6.12. Of course, the long exact sequence in polynomial cohomology can also be established directly via a diagram chasing argument.

6.13. Naturality with respect to the long exact sequence. In the sequel we will need the following observation concerning the map in linear cohomology induced by a measure equivalence:

Lemma 6.14. *Let (Ω, μ) be a (G, H) -coupling, and let $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$ be a short exact sequence of continuous Fréchet $G \times H$ -modules. Then for the isomorphisms $\chi^{\bullet}: H^{\bullet}(G, L_{loc}^2(X, -)) \rightarrow H^{\bullet}(H, L_{loc}^2(Y, -))$, as given by the Reciprocity Theorem 6.4, the induced diagram of long exact sequences*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^n(G, L_{loc}^2(X, \mathcal{E})) & \longrightarrow & H^n(G, L^2(X, \mathcal{F})) & \longrightarrow & H^n(G, L_{loc}^2(X, \mathcal{Q})) \longrightarrow \cdots \\ & & \downarrow \chi & & \downarrow \chi & & \downarrow \chi \\ \cdots & \longrightarrow & H^n(H, L_{loc}^2(Y, \mathcal{E})) & \longrightarrow & H^n(H, L^2(Y, \mathcal{F})) & \longrightarrow & H^n(H, L_{loc}^2(Y, \mathcal{Q})) \longrightarrow \cdots \end{array}$$

is commutative.

Proof. On the level of inhomogeneous cochains (in particular on inhomogeneous cocycles), χ is given by

$$(\chi^n \xi)(h_1, \dots, h_n)(t) = \chi_G(t) \cdot \xi(\chi_G(t)^{-1} \chi_G(h_1^{-1}t), \dots, \chi_G(t)^{-1} \chi_G(h_n^{-1}t))(\chi_G(t)^{-1}t).$$

The only statement needing proof is that χ^\bullet commutes with the connecting maps in cohomology. To this end let $\bar{\xi} \in Z^n(G, L_{loc}^2(\Omega, Q)^H)$ (recall that we freely identify $L_{loc}^2(\Omega, -)^H$ with $L_{loc}^2(X, -)$) and let $\xi \in L_{loc}^2(G^n, L_{loc}^2(\Omega, \mathcal{F})^H)$ be a lift of $\bar{\xi}$. Then $\chi^n(\xi)$ is a lift of $\chi^n(\bar{\xi})$, and clearly we have $\chi^{n+1}(d^n \xi) = d^n(\chi^n \xi)$. Thus if we let η denote the cocycle representative of $\delta(\bar{\xi})$ constructed using the usual diagram chase, given the lift ξ fixed previously, it follows that $\chi^{n+1}(\eta)$ is a cocycle representative of $\delta(\chi^n(\bar{\xi}))$, constructed as well via the usual diagram chase, given the lift $\chi^n(\xi)$ of $\chi^n(\bar{\xi})$. \square

7. REAL-VALUED POLYNOMIAL MAPS ON GROUPS

Let G be a lcsc group, \mathcal{E} a locally continuous G -module, and consider the standard relatively injective resolution

$$0 \longrightarrow \mathcal{E} \xrightarrow{d^{-1}} C(G, \mathcal{E}) \xrightarrow{d^0} C(G^2, \mathcal{E}) \xrightarrow{d^1} C(G^3, \mathcal{E}) \longrightarrow \dots$$

For any $\xi \in C(G^2, \mathcal{E})$ we denote by $\bar{\xi} \in C(G, \mathcal{E})$ the restriction $\bar{\xi}(g) = \xi(\mathbb{1}, g)$, $g \in G$. For functions $\xi: G \rightarrow \mathcal{E}$ we define also the *unitized difference operator* by

$$(\bar{\partial}_g \xi)(h) := (\partial_g \xi)(h) - (\partial_g \xi)(\mathbb{1}).$$

The following proposition generalizes the usual description of inhomogeneous 1-cocycles.

Proposition 7.1 (polynomial 1-cocycles). *Let G be a lcsc group, \mathcal{E} a continuous G -module, $d \in \mathbb{N}$. Then the map $\xi \mapsto \bar{\xi}$ is an isomorphism from $C(G^2, \mathcal{E})^{G(d)} \cap \ker(d^1)$ onto the space of functions $\eta \in C(G, \mathcal{E})$ satisfying either of the following equivalent conditions:*

- (i) $\eta(\mathbb{1}) = 0$ and for all g_1, \dots, g_d we have $(\bar{\partial}_{g_1} \circ \dots \circ \bar{\partial}_{g_d})(\eta) = 0$,
- (ii) $\eta(\mathbb{1}) = 0$ and $\eta \in C(G, \mathcal{E})^{G(d+1)}$.

Proof. The equivalence of (i) and (ii) follows easily by induction on d . Suppose that $\xi \in C(G^2, \mathcal{E})^{G(d)} \cap \ker(d^1)$ holds. Then we compute:

$$\begin{aligned} \overline{\partial_g(\xi)}(h) &= (\partial_g \xi)(\mathbb{1}, h) = g.\xi(g^{-1}, g^{-1}h) - \xi(\mathbb{1}, h) \\ &= g.\xi(\mathbb{1}, g^{-1}h) + g.\xi(g^{-1}, \mathbb{1}) - \xi(\mathbb{1}, h) \\ &= g.\xi(\mathbb{1}, g^{-1}h) - \xi(\mathbb{1}, h) - g.\xi(\mathbb{1}, g^{-1}) + \xi(\mathbb{1}, \mathbb{1}) \\ &= (\partial_g \bar{\xi})(h) - (\partial_g \bar{\xi})(\mathbb{1}), \end{aligned}$$

which then immediately implies that (i) holds for $\bar{\xi}$. For the other direction, suppose that (ii) holds for some $\eta \in C(G, \mathcal{E})$. Then, letting $\xi(g, h) = \eta(h) - \eta(g) \in C(G^2, \mathcal{E})$ and running the above computation in reverse we see that $\xi \in C(G^2, \mathcal{E})^{G(d)} \cap \ker(d^1)$. This shows surjectivity of $\xi \mapsto \bar{\xi}$. Injectivity is clear. \square

Example 7.2 (quadratic 1-cocycles). By the proposition, we may describe the inhomogeneous "quadratic" 1-cocycles $\xi: G \rightarrow \mathcal{E}$ as precisely those unital maps for which, for all $g, h \in G$, $(\partial_g \circ \partial_h)\xi$ is constant. Computing this we get

$$\begin{aligned} (\partial_g \circ \partial_h)(\xi)(k) &= gh.\xi((gh)^{-1}k) - g.\xi(g^{-1}k) - h.\xi(h^{-1}k) + \xi(k) \\ &= gh.\xi((gh)^{-1}) - g.\xi(g^{-1}) - h.\xi(h^{-1}), \end{aligned}$$

where the second equality follows by letting $k = \mathbb{1}$. This can be rewritten as

$$\xi(ghk) = \xi(gh) + g.\xi(hk) + ghg^{-1}.\xi(gk) - ghg^{-1}.\xi(g) - g.\xi(h) - gh.\xi(k), \quad g, h, k \in G.$$

In the remainder of this section we focus on polynomial 1-cohomology with trivial coefficients $\mathcal{E} = \mathbb{R}$. In this case, the maps $G \rightarrow \mathbb{R}$ satisfying the conditions (i),(ii) in the previous proposition are called polynomial maps on G , see e.g. [21]. The (abstract) notion of polynomial maps on groups seems to go back at least to [3, 20]. We now formally define the space of polynomial maps:

Definition 7.3 (polynomial maps). Let G be a lcsc group and let $\xi \in C(G, \mathbb{R})$. We say that ξ is a polynomial of degree at most $d \in \mathbb{Z}_*$ if for all $g_1, \dots, g_{d+1} \in G$ we have

$$(7.1) \quad (\partial_{g_1} \circ \dots \circ \partial_{g_{d+1}})(\xi) \equiv 0.$$

The degree $\deg \xi$ of a polynomial map ξ is the smallest number d such that ξ satisfies (7.1) for all $g_1, \dots, g_{d+1} \in G$, that is, the smallest $d \in \mathbb{Z}_*$ such that $\xi \in C(G, \mathbb{R})^{G(d+1)}$, with the convention that the degree of the constant zero polynomial is $\deg 0 = -\infty$. We denote for convenience $\text{Pol}_d(G) := C(G, \mathbb{R})^{G(d+1)}$ and $\text{Pol}(G) = \cup_d \text{Pol}_d(G)$.

Remark 7.4. The following observation will prove useful: let us define for any function $\xi: G \rightarrow \mathbb{R}$ the right-difference

$$(\mathfrak{G}_g \xi)(h) := \xi(hg) - \xi(h).$$

Then by induction on d it is easy to see that $\xi \in C(G, \mathbb{R})$ is a polynomial map of degree at most d if and only if for all $g_1, \dots, g_{d+1} \in G$ we have

$$(\mathfrak{G}_{g_1} \circ \dots \circ \mathfrak{G}_{g_{d+1}})(\xi) \equiv 0.$$

That is, left-polynomials and right-polynomials coincide [21, Corollary 2.13], and the degree of a polynomial map ξ into \mathbb{R} coincides for left- and right-differentiation. For a more general result, see [21, Proposition 3.16].

Proposition 7.1 thus says, in particular, that with trivial coefficients, the space of polynomial degree d , homogeneous 1-cocycles in the standard resolution is isomorphic to the space of (continuous) unital polynomial maps of degree at most d . The following proposition describes (for trivial coefficients) precisely the polynomial 1-coboundaries:

Proposition 7.5. *Let G be a lcsc group and let $d \in \mathbb{N}$. Under the map $\xi \mapsto \bar{\xi}$ as above, the image $d^0(C(G, \mathbb{R})^{G(d)})$ maps bijectively onto the space of (continuous) unital polynomial maps of degree at most $d - 1$, that is*

$$H_{(d)}^1(G, \mathbb{R}) \cong \text{Pol}_d(G) / \text{Pol}_{d-1}(G).$$

Further, the natural G -action on $H_{(d)}^1(G, \mathbb{R})$ is trivial.

Proof. Suppose that $\xi \in C(G^2, \mathbb{R})^{G(d)} \cap \text{im}(d^0)$, say $\xi = d^0(\eta)$. Then applying d times the differential, we get that $0 = (\partial_{g_1} \circ \dots \circ \partial_{g_d})(\xi) = d^0((\partial_{g_1} \circ \dots \circ \partial_{g_d})(\eta))$. Thus $(\partial_{g_1} \circ \dots \circ \partial_{g_d})(\eta)$ is constant, from which it follows that $\eta \in C(G, \mathbb{R})^{G(d+1)}$. The statement follows readily from this. \square

Proposition 7.6. *Let G be a group and $\xi: G \rightarrow \mathbb{R}$ be a (continuous) polynomial map of degree $\deg \xi = d \geq 1$. Then for every $s \in G$, the map $\varphi_\xi: g \mapsto (\partial_g \xi)(s)$ is a (continuous) polynomial map of degree $\deg \varphi_\xi = (d-1)+1$. Analogously for $g \mapsto (\mathfrak{G}_g \xi)(s)$.*

Proof. By a direct computation, one verifies that the differential satisfies, for any function $\xi: G \rightarrow \mathbb{R}$

$$\partial_{gh} \xi = (\partial_g \circ \partial_h)(\xi) + \partial_g \xi + \partial_h \xi, \quad g, h \in G.$$

Thus for any $h \in G$ we have

$$\begin{aligned} (\mathbb{G}_h \varphi_\xi)(g) &= \varphi_\xi(gh) - \varphi_\xi(g) \\ &= (\partial_g \circ \partial_h)(\xi)(s) + (\partial_h \xi)(s) \\ &= \varphi_{\partial_h \xi}(g) + (\partial_h \xi)(s). \end{aligned}$$

The proposition follows by induction on $\deg \xi$. \square

Let G be a lsc group and let $\xi, \eta: G \rightarrow \mathbb{R}$ be continuous polynomial maps on G . Then it is easy to see by induction on $\deg \xi + \deg \eta$ that the pointwise product $\xi \cdot \eta: g \mapsto \xi(g) \cdot \eta(g)$ is again a (continuous) polynomial map with $\deg(\xi \cdot \eta) = \deg \xi + \deg \eta$. Indeed, we have

$$(7.2) \quad \begin{aligned} (\mathbb{G}_g(\xi \cdot \eta))(h) &= \xi(hg) \cdot (\mathbb{G}_g \eta)(h) + (\mathbb{G}_g \xi)(h) \cdot \eta(h), \quad \text{and} \\ (\partial_g(\xi \cdot \eta))(h) &= \xi(g^{-1}h) \cdot (\partial_g \eta)(h) + (\partial_g \xi)(h) \cdot \eta(h), \end{aligned}$$

from which the inequality ' \leq ' is immediate, by induction on $\deg \xi + \deg \eta$. The opposite inequality follows by Lemma 7.7 below after observing that, given any continuous polynomial $\xi: G \rightarrow \mathbb{R}$ of degree d , say, there is a finitely generated free nilpotent group F such that the pull-back of ξ under some homomorphism $F \rightarrow G/G_{[d+1]}$ has degree d as well.

In particular the multiplication map induces a linear map

$$H_{(d)}^1(G, \mathbb{R}) \otimes H_{(d')}^1(G, \mathbb{R}) \rightarrow H_{(d+d')}^1(G, \mathbb{R})$$

for each pair $(d, d') \in \mathbb{N}_0^2$.

Let G be a Mal'cev group and let $(g_{i,j})$ be a Mal'cev basis of G . Then for each pair (i_0, j_0) we consider the map $\zeta_{g_{i_0, j_0}}: G \rightarrow \mathbb{R}$ given by

$$(7.3) \quad \zeta_{g_{i_0, j_0}} \left(\prod_{(i,j) \in \mathbf{B}_{\mathbf{rk}(G)}} g_{i,j}^{t_{i,j}} \right) = t_{i_0, j_0},$$

where for all i, j and all $t_{i,j} \in Z_{i,j}$, the components of the Mal'cev coordinate system in G associated with $(g_{i,j})$, see Section 2. In general, for a residually Mal'cev group G we define $\zeta_{g_{i_0, j_0}}$ as the composition

$$G \longrightarrow G/G_{i_0+1} \xrightarrow{\bar{\zeta}_{g_{i_0, j_0}}} \mathbb{R}$$

where G_{i_0+1} is the $i_0 + 1$ -th term in the Mal'cev central series and $\bar{\zeta}_{g_{i_0, j_0}}$ is as in (7.3) for G/G_{i_0+1} . Finally, we will need the following notation: for any multi-index $\mathbf{d} \in \mathbf{D}_{d, \dim(G)}$ we define

$$\zeta_{\mathbf{d}} := \prod_{(i,j) \in \mathbf{B}_{\mathbf{rk}(G)}} \zeta_{g_{i,j}}(-)^{d_{i,j}}.$$

Lemma 7.7. *Let G be a residually Mal'cev group and let $(g_{i,j})$ be a Mal'cev basis. Then for all (i_0, j_0) the map $\zeta_{g_{i_0, j_0}}$ as above is a polynomial map of degree $\deg \zeta_{g_{i_0, j_0}} = i_0$. Furthermore the set $\{\zeta_{\mathbf{d}} \mid \mathbf{d} \in \mathbf{D}_{d, \dim(G)}\}$ is a linear basis of $\text{Pol}_d(G)$.*

Proof. Given a residually Mal'cev group G we observe first that for each d the canonical projection map $\kappa: G \rightarrow G/G_{d+1}$ induces an isomorphism on polynomials $\kappa^*: \text{Pol}_d(G/G_{d+1}) \rightarrow \text{Pol}_d(G)$. Indeed, injectivity is trivial and surjectivity follows as in [21, Proposition 2.15] and noting that any polynomial map $K \rightarrow \mathbb{R}$ on a compact group K is constant. Thus we may assume that G is a Mal'cev group, and then that in fact (By the Shapiro Lemma [17] and

using property H_T , see Corollary 5.7, and Proposition 6.8) G is csc nilpotent: indeed, let L be the Mal'cev completion of G ; the results just cited then yield, inductively, for any $d \in \mathbb{N}$ that

$$\begin{aligned} H_{(d)}^1(G, \mathbb{R}) &= H^1(G, \text{Pol}_{d-1}(G)) \\ &= H^1(G, \text{Pol}_{d-1}(L)) \\ &= H^1(G, L_{\text{loc}}^2(L)^{L(d-1)}) \\ &= H_{(d)}^1(L, L_{\text{loc}}^2(L)^G) \\ &= H_{(d)}^1(L, L^2(L/G)) \\ &= H_{(d)}^1(L, \mathbb{R}). \end{aligned}$$

Note that property H_T gives us $H^1(L, L_0^2(L/G)) = 0$ since, by the previous equalities and the fact that G is cohomologically finite dimensional, we already know that it is Hausdorff.

The first part of the lemma follows from [21, Theorem 3.2]. Indeed, proceeding by induction on $\dim \mathfrak{g}$ we first show that for $z = g_{\text{cl}(G), j_0}$, the map $h \mapsto z^{\zeta_z(h)}$ is an *lc-polynomial* [21] of lc-degree at most $(1, \dots, \text{cl}(G))$. But this is clear since we can write, for every $h \in G$

$$(7.4) \quad z^{\zeta_z(h)} = h \cdot \left(\prod_{(i,j) \neq (\text{cl}(G), j_0)} g_{i,j}^{\zeta_{g_{i,j}}(h)} \right)^{-1},$$

and each factor $h \mapsto g_{i,j}^{\zeta_{g_{i,j}}(h)}$, $(i,j) \neq (\text{cl}(G), j_0)$ is an lc-polynomial map of lc-degree at most $(1, 2, \dots, \text{cl}(G))$, by the induction hypothesis, since they factor through $G/z^{\mathbb{R}}$. Hence $h \mapsto z^{\zeta_z(h)} \in z^{\mathbb{Z}} \leq G$ is an lc-polynomial as well, and so ζ_z is a polynomial map of degree at most $\text{cl}(G)$. To see that $\deg \zeta_z = \text{cl}(G)$, observe that $(\mathfrak{G}_{g_{1,j_1}} \zeta_z)(g_{\text{cl}(G)-1, j_2}) \neq 0$ for some g_{1,j_1} and $g_{\text{cl}(G)-1, j_2}$ with $\zeta_z([g_{1,j_1}, g_{\text{cl}(G)-1, j_2}]) \neq 0$, whence $\deg(\mathfrak{G}_{g_{1,j_1}} \zeta_z) \geq \text{cl}(G) - 1$.

To see the second part of the lemma we proceed by induction on $\dim \mathfrak{g}$, the case $\dim \mathfrak{g} = 1$ being clear. Fix a basis element $z = g_{\text{cl}(G), j_0}$ for some j_0 , and let $\xi \in \text{Pol}(G)$ be given. If $\mathfrak{G}_z \xi \equiv 0$ then ξ descends to a polynomial map $\bar{\xi}: G/z^{\mathbb{R}}$ and so the claim follows directly by the induction hypothesis. Next we observe that, by (7.2)

$$(7.5) \quad \mathfrak{G}_z((\mathfrak{G}_z \xi) \cdot \zeta_z - \xi) = (\mathfrak{G}_z \circ \mathfrak{G}_z)(\xi) \cdot (1 - \zeta_z),$$

whence the claim follows directly in the case $\mathfrak{G}_z^{(2)} \xi \equiv 0$. To deduce the general case, note that for any product $\eta := \eta_1 \cdot \eta_2$ of maps $\eta_i: G \rightarrow \mathbb{R}$ such that $\mathfrak{G}_z^{(m_i)} \eta_i \equiv 0$, $i = 1, 2$, we have by induction and 7.2 that $\mathfrak{G}_z^{(\max\{m_1, m_2\})} \eta \equiv 0$; the general case follows now inductively: combining equation (7.5) with this observation, we conclude by the induction hypothesis that $(\mathfrak{G}_z^{(n-1)} \xi) \cdot \zeta_z - \xi$ can be written as a linear combination of pointwise products of $\zeta_{g_{i,j}}, g_{i,j} \neq z$, which proves that the $\zeta_{\mathbf{d}}$ span $\text{Pol}_d(G)$. The linear independence is clear since the $\zeta_{\mathbf{d}}$ define, by pulling back through the Mal'cev coordinates, mutually linearly independent polynomials on $\mathbb{R}^{\dim \mathfrak{g}}$. \square

We will need also the following uniqueness results for polynomial maps into \mathbb{R} .

Lemma 7.8. *Let G be a (residually) Mal'cev group and let $(g_{i,j})$ be a Mal'cev basis of G . Denote $c := \text{rk } G$ and let G_0 be the (not necessarily closed) subgroup of G generated (as an abstract group) by $S := \{g_{1,1}, \dots, g_{1,c}\}$. Finally, denote by $S^{\leq d}$ the set of words in G_0 on S ,*

of length at most $d \in \mathbb{N}$. Then for any $d \in \mathbb{N}$ and all $\xi, \eta \in \text{Pol}_d(G)$ we have

$$\xi = \eta \Leftrightarrow \xi|_{S^{\leq d}} = \eta|_{S^{\leq d}}.$$

In particular, any polynomial map in $\text{Pol}(G)$ is determined uniquely by its values on G_0 .

Proof. As in [21, Proposition 1.15], we see that if $\xi|_{S^{\leq d}} = \eta|_{S^{\leq d}}$ then $\xi|_{G_0} = \eta|_{G_0}$, so the lemma will follow if we show that $\xi \in \text{Pol}(G)$ vanishes on G if it vanishes on G_0 .

This follows from Lemma 7.7, for instance by the following argument: given $\xi \in \text{Pol}(G)$ such that $\xi|_{G_0} \equiv 0$, we can write $\xi = \sum_{i=1}^n \eta_i \cdot \zeta_z^i$ where $\zeta_z \eta_i \equiv 0$ for all i , where $z \in G_0 \cap Z(G)$. By induction on $\dim_{\mathbb{R}} \mathfrak{g}$, we conclude that $\eta_i \equiv 0$ for all i from which the statement follows. \square

Remark 7.9. In general, for a lcsc group G and an inhomogeneous polynomial 1-cocycle $\xi: G \rightarrow \mathcal{E}$ of degree $d \in \mathbb{N}$, it is not hard to show by induction on d that ξ is entirely determined by its restriction to $S^{\leq d}$ where S is a generating set of G .

Lemma 7.10. Let G be a (residually) Mal'cev group, $(g_{i,j})$ a Mal'cev basis, and let $\xi, \eta \in \text{Pol}(G)$. If $\xi(1) = \eta(1)$ and $\zeta_{g_{1,j}} \xi = \zeta_{g_{1,j}} \eta$ for all $j = 1, \dots, \text{rk}(G)$, then $\xi = \eta$.

The lemma follows from the previous lemma, by showing that $\xi(g) = \eta(g)$ for all $g \in G_0$, by induction on word-length. Here is an alternative argument:

Proof. It is easy to see that, for any $f \in C(G, \mathbb{R})$ the function $g \mapsto \zeta_g f$ satisfies the 1-cocycle identity. Hence we conclude that $\zeta_g(\xi - \eta) = 0$ for all $g \in G_0$, the subgroup of G generated by the $g_{1,j}$. Since $g \mapsto \zeta_g(\xi - \eta)(h)$ is itself a continuous polynomial map on G for any $h \in G$ (Proposition 7.6), the statement follows now by the previous lemma. \square

Using Lemma 7.7, we can now give a sharper estimate on the degree of $\partial_g \xi$ for a polynomial map ξ .

Definition 7.11 (degree wrt. a central series). Let G be a (lcsc) group and \mathcal{G} a central series in G with trivial intersection. For every $g \in G$ we define the degree $\deg_{\mathcal{G}} g$ of g with respect to the central series \mathcal{G} by

$$\deg_{\mathcal{G}} g = \max\{i \mid g \in G_i, g \notin G_{i+1}\}.$$

When G is a residually Mal'cev group, the degree $\deg g$ of an element g in G , will refer to the degree wrt. the Mal'cev central series, unless explicitly stated otherwise. For a subset $K \subseteq G$ we set $\deg_{\mathcal{G}} K = \min_{g \in K} \deg_{\mathcal{G}} g$.

Lemma 7.12. Let G be a (residually) Mal'cev group and let $\xi \in \text{Pol}(G)$. Then for every $g \in G$ we have $\deg \partial_g \xi \leq \deg \xi - \deg g$ and analogously for $\zeta_g \xi$.

Proof. By Lemma 7.7 and (7.2), it is sufficient to prove the claim for all $\zeta_{g_{i,j}}$. This follows by induction on $\dim \mathfrak{g}$, using (7.4). \square

Lemma 7.13. Let G be a free nilpotent group of class $d \in \mathbb{N}$, and let $g_i = g_{1,i}, i = 1, \dots, n := \text{rk}(G)$ be a set of generators of G . Then for any $d' < d$ and any set $\xi_i: G \rightarrow \mathbb{R}, i = 1, \dots, n$ of polynomial maps ξ_i of degree at most d' , there is a polynomial map ξ of degree $\deg \xi \leq d'$ such that $\partial_{g_i} \xi = \xi_i$ for all i .

Proof. It is not hard to give a direct proof of this proposition, analogous to the universal property for the free group with, respect to polynomial maps, proved in [21]. A direct

argument can also be given based on Lemma 7.7. Here we give a third proof. Let's write it out in detail:

By induction on d we may assume that $\deg \xi_i = d - 1$ for some i , and all we have to show is that there is a polynomial ξ such that $\partial_{g_i} \xi = \xi_i \bmod \text{Pol}_{d-2}(G)$ for all i . To this end, let $F = \langle f_1, \dots, f_n \rangle$ be the free group on n generators and $\pi: F \rightarrow G: f_i \mapsto g_i$ the induced homomorphism. On F we consider the group homomorphism $\eta: F \rightarrow H_{(d-1)}^1(F, \mathbb{R}): f_i \mapsto \xi_i$. Then we observe that in the long exact sequence in cohomology (for F) induced by $\text{Pol}_{d-2}(F) \rightarrow \text{Pol}_{d-1}(F) \rightarrow H_{(d-1)}^1(F, \mathbb{R})$, we have

$$\dots \longrightarrow H^1(F, \text{Pol}_{d-2}(F)) \xrightarrow{0} H^1(F, \text{Pol}_{d-1}(F)) \xrightarrow{\cong} H^1(F, H_{(d-1)}^1(F, \mathbb{R})) \longrightarrow 0 \longrightarrow \dots,$$

whence there exists a polynomial map $\xi' \in \text{Pol}_d(F)$ such that $\partial_{f_i} \xi' = \eta(f_i)$ in $H_{(d-1)}^1(F, \mathbb{R})$, for all i . Noting that π induces an isomorphism $\pi^*: \text{Pol}_d(G) \rightarrow \text{Pol}_d(F)$, cf. Lemma 7.7, this finishes the proof. \square

8. THE ALGEBRA OF POLYNOMIAL MAPS

The space of polynomial maps $\text{Pol}_d(G)$ introduced in the previous section should be seen, as mentioned in the introduction, as containing certain "d'th order" dual structure. In particular, $\text{Pol}_1(G)$, being essentially (that is, up to addition of some constant) the space of continuous group homomorphisms into \mathbb{R} , contains very precise information about the (torsion-free part of the) abelianization of G . In this spirit, Proposition 8.9 below is the key observation which will allow us to lift isomorphisms in polynomial cohomology induced by a uniform measure equivalence, to an isomorphism of groups in Theorem B. This reduces to a standard duality trick when $d = 1$, where all statements in this section are essentially trivial.

Lemma 8.1. *Let G_1, G_2 be residually Mal'cev groups and denote the canonical projections $\pi_i: G_1 \times G_2 \rightarrow G_i, i = 1, 2$. Then the pull-backs π_i^* induce embeddings $\pi_i^*: \text{Pol}(G_i) \rightarrow \text{Pol}(G_1 \times G_2), i = 1, 2$, and we have with this identification an isomorphism $\text{Pol}(G_1 \times G_2) \cong \text{Pol}(G_1) \otimes \text{Pol}(G_2)$ such that $(\xi_1 \otimes \xi_2)(g_1, g_2) = \xi_1(g_1) \cdot \xi_2(g_2)$. Furthermore, this isomorphism respects the grading given by the polynomial degree, that is, for any $\xi_i \in \text{Pol}(G_i), i = 1, 2$ we have $\deg(\xi_1 \otimes \xi_2) = \deg \xi_1 + \deg \xi_2$*

Proof. This all follows directly from Lemma 7.7. \square

In particular, the previous lemma shows that, given any linear map $\Psi: \text{Pol}(G) \rightarrow \text{Pol}(H)$, where G, H are residually Mal'cev groups, such that $\deg \Psi(\xi) \leq \deg \xi$ for all $\xi \in \text{Pol}(G)$, we get naturally an induced map $\Psi \otimes \Psi: \text{Pol}(G \times G) \rightarrow \text{Pol}(H \times H)$ given by $\Psi(\xi \otimes \eta) = \Psi(\xi) \otimes \Psi(\eta)$ for all $\xi, \eta \in \text{Pol}(G)$ such that $\deg(\Psi \otimes \Psi)(\xi') \leq \deg \xi'$.

Definition 8.2 (degree-preserving maps). Let G, H be lcsc groups. We will say that a linear map $\Psi: \text{Pol}(G) \rightarrow \text{Pol}(H)$ is *degree-preserving* if $\deg \Psi(\xi) \leq \deg \xi$ for all $\xi \in \text{Pol}(G)$, and *properly degree-preserving* if equality holds.

Definition 8.3 (strongly unital maps). We say that a linear map $\Psi: \text{Pol}(G) \rightarrow \text{Pol}(H)$ is *strongly unital* if it is unital, and if $\Psi(\xi)(1) = 0$ whenever $\xi(1) = 0$.

Remark 8.4. Occasionally we will use the terms '(properly) degree-preserving' and 'strongly unital', with the obvious meanings, for maps $\Psi: \text{Pol}_d(G) \rightarrow \text{Pol}_d(H)$ for some given d .

In the sequel we will denote by $m: G \times G \rightarrow G$ the multiplication map on an arbitrary group G . Furthermore, $\tilde{m}: G \times G \rightarrow G$ denotes the map $\tilde{m}(g, h) = gh^{-1}$ and $\text{inv}: G \rightarrow G$ the inversion map.

Definition 8.5 (co-multiplicativity). Let G, H be residually Mal'cev groups. We say that a linear map $\Psi: \text{Pol}(G) \rightarrow \text{Pol}(H)$ is *co-multiplicative* if the following diagram commutes:

$$\begin{array}{ccc} \text{Pol}(G) & \xrightarrow{m^*} & \text{Pol}(G^2) \\ \Psi \downarrow & & \downarrow \Psi \otimes \Psi \\ \text{Pol}(H) & \xrightarrow{m^*} & \text{Pol}(H^2) \end{array} .$$

For (degree-preserving) $\Psi: \text{Pol}_d(G) \rightarrow \text{Pol}_d(H)$ we will say that Ψ is co-multiplicative in degree d if the diagram above, replacing $\text{Pol}(-)$ by $\text{Pol}_d(-)$ is commutative.

Remark 8.6. Let G be a Mal'cev group. We note that, by [21], m^* is a well-defined (that is, $m^*\xi$ is a polynomial for every polynomial ξ) properly degree-preserving map. Indeed, we claim that multiplication $m: G \times G \rightarrow G$ is a polynomial map of lc-degree [21, Section 3] $\text{lc-deg } m = (1, \dots, \text{cl}(G))$. To see this, let $\pi_i: G \times G \rightarrow G, i = 1, 2$ denote the projections on the first, respectively second factor. Then $m(g) = \pi_1(g) \cdot \pi_2(g)$ is a pointwise product of homomorphisms, so the claim follows by [21, Theorem 3.2]. From this it follows readily that $m^*\xi$ is a polynomial with $\deg m^*\xi = \deg \xi$ for all $\xi \in \text{Pol}(G)$. Thus all maps in the diagram in Definition 8.5 are well-defined when G has finite length, and the general case follows since any $\xi \in \text{Pol}_d(G)$ factors through $G/G_{\deg \xi + 1}$. In fact we observe that, by Lemma 7.7, the pull-back has the form

$$(8.1) \quad m^*(\xi) = \xi \otimes \mathbb{1} + \left(\sum_i \xi_i \otimes \xi'_i \right) + \mathbb{1} \otimes \xi$$

where each term $\xi_i \otimes \xi'_i$ can be taken to be a simple tensor consisting of basis elements as in 7.7. Then for $\xi = \zeta_z$ a simple basis element, it follows directly from Lemma 7.7 that $\deg \xi_i + \deg \xi'_i \leq \deg \xi$ for all i , and $\deg \xi_i, \deg \xi'_i < \deg \xi$.

Remark 8.7. One verifies easily that a strongly unital degree-preserving algebra homomorphism $\Psi: \text{Pol}(G) \rightarrow \text{Pol}(H)$ of residually Mal'cev groups G, H , is co-multiplicative if and only if the following diagram commutes.

$$(8.2) \quad \begin{array}{ccc} \text{Pol}(G) & \xrightarrow{\tilde{m}^*} & \text{Pol}(G^2) \\ \Psi \downarrow & & \downarrow \Psi \otimes \Psi \\ \text{Pol}(H) & \xrightarrow{\tilde{m}^*} & \text{Pol}(H^2) \end{array} .$$

Indeed, we have $m^* = (\mathbb{1} \otimes \text{inv}) \circ \tilde{m}^*$, so that it is sufficient, for the 'if' direction, to see that, say, if diagram (8.2) is commutative, then so is the following diagram:

$$\begin{array}{ccc} \text{Pol}(G) & \xrightarrow{\text{inv}^*} & \text{Pol}(G) \\ \Psi \downarrow & & \downarrow \Psi \\ \text{Pol}(H) & \xrightarrow{\text{inv}^*} & \text{Pol}(H) \end{array} .$$

This will imply that Ψ is co-multiplicative. To see the claim, observe that for any unital polynomial map ξ and all $h \in H$, we have (using (8.2) in the second equality, and the analogue of (8.1) for \tilde{m} in the first and third equalities)

$$\begin{aligned}\Psi(\xi)(h^{-1}) &= (\Psi \otimes \Psi)(\tilde{m}^*\xi)(h^{-1}, \mathbf{1}) \\ &= (\Psi \otimes \Psi)(\tilde{m}^*\xi)(\mathbf{1}, h) \\ &= \Psi(\tilde{\xi})(h).\end{aligned}$$

To see the other direction, we show again that, given a co-multiplicative Ψ , it follows that Ψ commutes with the inverse operator: (to get the third equality, compute $(m^*\tilde{\xi})(h, k)$)

$$\begin{aligned}\Psi(\tilde{\xi})(h) &= m^*(\Psi(\tilde{\xi}))(h, \mathbf{1}) \\ &= (\Psi \otimes \Psi)(m^*\tilde{\xi})(h, \mathbf{1}) \\ &= (\Psi \otimes \Psi)(m^*\xi)(\mathbf{1}, h^{-1}) \\ &= \Psi(\xi)(h^{-1}).\end{aligned}$$

Lemma 8.8. *Let G, H be residually Mal'cev groups and let $\Psi_k: \text{Pol}(G) \rightarrow \text{Pol}(H)$, $k = 1, 2$, be strongly unital, co-multiplicative, degree-preserving algebra homomorphisms. Let $(h_{i,j})$ be a Mal'cev basis for H , and $(g_{i,j})$ be a Mal'cev basis for G . Suppose that for all $\ell = 1, \dots, \text{rk}(H)$ and all i, j we have*

$$(8.3) \quad (\Psi_1 \zeta_{g_{i,j}})(h_{1,\ell}) = (\Psi_2 \zeta_{g_{i,j}})(h_{1,\ell}).$$

Then $\Psi_1 = \Psi_2$.

Proof. We show that $\Psi_1(\xi) = \Psi_2(\xi)$ by induction on $d := \deg \xi$. The case $d = 1$ follows directly from the hypotheses, in particular using that Ψ_i are algebra homomorphisms and Lemma 7.7. Let $d > 1$ be given. Suppose that $(\Psi_1 \xi)(h) = (\Psi_2 \xi)(h)$ and $(\Psi_1 \xi)(k) = (\Psi_2 \xi)(k)$ for some $h, k \in H$. Then by this and the induction hypothesis we get:

$$\begin{aligned}\Psi_1(\xi)(hk) &= m^*(\Psi_1(\xi))(h, k) \\ &= (\Psi_1 \otimes \Psi_1)(m^*(\xi))(h, k) \\ &= \sum_i \Psi_1(\xi_i)(h) \cdot \Psi_1(\xi'_i)(k) + \Psi_1(\xi)(h) + \Psi_1(\xi)(k) \\ &= \sum_i \Psi_2(\xi_i)(h) \cdot \Psi_2(\xi'_i)(k) + \Psi_2(\xi)(h) + \Psi_2(\xi)(k) \\ &= (\Psi_2 \otimes \Psi_2)(m^*(\xi))(h, k) \\ &= \Psi_2(\xi)(hk).\end{aligned}$$

Using this computation repeatedly, and by the assumption in (8.3) it follows that $(\Psi_1 \xi)(h) = (\Psi_2 \xi)(h)$ for all $h = h_{1,\ell}^n$ with $n \geq 0$, whence for all $h = h_{1,\ell}^n$ with $n \in \mathbb{Z}$, whence (by the previous computation) for all h in the subgroup H_0 of H consisting of words in the $h_{1,\ell}$. By Lemma 7.8 it follows that $\Psi_1 \xi = \Psi_2 \xi$. This completes the proof. \square

Observe that if $\varphi: H \rightarrow G$ is a homomorphism then it induces a unital, co-multiplicative, degree-preserving algebra homomorphism $\varphi^*: \text{Pol}(G) \rightarrow \text{Pol}(H)$. The next result gives a converse to this, in the spirit that $\text{Pol}(G)$ acts as a "total" dual space of G .

Proposition 8.9. *Let G, H be residually Mal'cev groups. Suppose that $\Psi: \text{Pol}(G) \rightarrow \text{Pol}(H)$ is a strongly unital, co-multiplicative, degree-preserving algebra homomorphism of the polynomial algebras. Then there is a unique continuous group homomorphism $\varphi: H \otimes \mathbb{R} \rightarrow G \otimes \mathbb{R}$ of the Mal'cev completions, such that Ψ is induced by φ . Further, φ is an isomorphism if and only if Ψ is a properly degree-preserving isomorphism.*

Observe that any degree-preserving isomorphism as in the statement is automatically properly degree-preserving; this follows by induction on the degree, using (8.1).

Proof. First, let us suppose that G, H have finite length, that is, they are Mal'cev groups. Fix Mal'cev bases $\{g_{i,j}\}$ respectively $\{h_{i,j}\}$. Let F be the free nilpotent Lie group of class $\text{cl}(F) = \max\{\text{cl}(G), \text{cl}(H)\}$, and with $\text{rk}(H)$ generators $f_{1,1}, \dots, f_{1,\text{rk}(H)}$. Then there is a unique homomorphism $\varphi_H: F \rightarrow H \otimes \mathbb{R}$ given by $\varphi_H(f_{1,\ell}) = h_{1,\ell}$, $\ell = 1, \dots, \text{rk}(H)$. Further, we define a homomorphism $\varphi_G: F \rightarrow G \otimes \mathbb{R}$ by

$$\varphi_G(f_{1,\ell}) = \prod_{(i,j) \in \mathbf{B}_{\text{rk}(G)}} g_{i,j}^{(\Psi \zeta_{g_{i,j}})(h_{1,\ell})}.$$

By Lemma 8.8 we conclude that the following diagram commutes:

$$\begin{array}{ccc} \text{Pol}(G) & \xrightarrow{\Psi} & \text{Pol}(H) \\ \varphi_G^* \searrow & & \swarrow \varphi_H^* \\ & \text{Pol}(F) & \end{array}$$

Thus, if $f \in F$ is an element in $\ker \varphi_H$, we have $\varphi_G^*(\zeta)(f) = 0$ for all unital $\zeta \in \text{Pol}(G \otimes \mathbb{R}) = \text{Pol}(G)$. It follows, since real-valued polynomials separate points on $G \otimes \mathbb{R}$, that $\varphi_G(f) = \mathbb{1}$. That is, we have shown that the homomorphism $\varphi_G: F \rightarrow G \otimes \mathbb{R}$ factors through $H \otimes \mathbb{R}$; let us denote the induced homomorphism $\varphi: H \otimes \mathbb{R} \rightarrow G \otimes \mathbb{R}$. We conclude immediately, again by Lemma 8.8 that $\varphi^* = \Psi$, which also gives the uniqueness of φ . Further, if Ψ is an isomorphism then it follows that φ is an isomorphism as well: indeed, φ has cocompact image since Ψ induces in particular an isomorphism of the abelianizations of $H \otimes \mathbb{R}$ and $G \otimes \mathbb{R}$, whence by simply connectedness, it is surjective. Injectivity follows since real-valued polynomials separate points on $H \otimes \mathbb{R}$.

Finally we note that the Proposition follows for residually Mal'cev groups in general by a straight-forward argument passing to the projective limits, using the uniqueness of φ , and, for the isomorphism statement, the fact that Ψ is an isomorphism if and only if $\Psi|_{\text{Pol}_d(G)}$ is an isomorphism for every $d \in \mathbb{N}$. \square

Remark 8.10. In the sequel we will use the following observation, which follows directly from the proof of the previous proposition: With hypotheses as in the proposition, assume further that $\text{cl}(G) = \text{cl}(H) := d$. Then it is sufficient that Ψ is a (strongly unital, linear) co-multiplicative degree-preserving isomorphism $\Psi: \text{Pol}_d(G) \rightarrow \text{Pol}_d(H)$, such that $\Psi(\xi \cdot \eta) = \Psi(\xi) \cdot \Psi(\eta)$ whenever $\deg \xi + \deg \eta \leq d$, $\xi, \eta \in \text{Pol}(G)$.

9. SHALOM'S INDUCTION MAP IN POLYNOMIAL COHOMOLOGY

Let G, H be lscu groups and suppose that they admit a *uniform* coupling (Ω, μ) . For any polynomial map $\zeta \in \text{Pol}_d(G)$, we set

$$(9.1) \quad (\omega_G^* \zeta)(h) = \int_Y \zeta(\tilde{\omega}_G(h, y)) d\mu_Y(y).$$

Shalom showed (for countable discrete groups, and remarking also that essentially the same proof works for general lcscu groups) in [30] that, if G, H have property $H_T(1)$ (for Hilbert spaces) then ω_G^* induces an isomorphism in cohomology $H^1(G, \mathbb{R}) \xrightarrow{\cong} H^1(H, \mathbb{R})$. The following example shows that ω_G^* does not induce directly a map in polynomial cohomology.

Example 9.1. Let $G := \mathbb{Z}, H := \mathbb{R}$ and let $(\Omega, \mu) = (\mathbb{R}, \lambda)$ be the obvious uniform (G, H) -coupling. Then $Y = \mathbb{R}/\mathbb{Z}$ and, denoting by $\sigma: Y \rightarrow \mathbb{R}$ the canonical section, the cocycle ω_G is given by

$$\omega_G(h, y) = \lfloor h + \sigma(y) \rfloor.$$

It is then easy to see that for the polynomial map $\zeta: G \ni n \mapsto n^2 \in \mathbb{R}$, we get

$$(\omega_G^* \zeta)(n + t) = (1 - t) \cdot n^2 + t \cdot (n + 1)^2, \quad n \in \mathbb{Z}, t \in [0, 1).$$

Thus, while ω_G^* in the previous example does not induce a map in quadratic cohomology directly, that is, via. a map on cocycles, it does so up to perturbations of at most linear magnitude.

Theorem 9.2. *Let G, H be cohomologically finite dimensional lcscu groups with property H_T^0 , and suppose that (Ω, μ) is a uniform, ergodic (G, H) -coupling. Then the following hold:*

(i) *For each $n \in \mathbb{N}$ and every $d \in \mathbb{N}$ there is an isomorphism*

$$\Psi_{(d)}^n: H_{(d)}^n(G, \mathbb{R}) \xrightarrow{\cong} H_{(d)}^n(H, \mathbb{R}).$$

(ii) *Suppose that $\bar{G} := G/G_2 \cong \mathbb{R}^k$ for some $k \in \mathbb{N}_0$ and denote by $\psi_1: H/H_2 \rightarrow \bar{G}$ the homomorphism (pre-)dual to $\Psi_{(1)}^1$. Then for every $d \in \mathbb{N}$ there is a commutative diagram*

$$\begin{array}{ccc} H_{(d)}^1(G, \mathbb{R}) & \xlongequal{\quad} & H^1(G, \text{Pol}_{d-1}(G)) \longrightarrow H^1(G, H_{(d-1)}^1(G, \mathbb{R})) \\ \Psi_{(d)}^1 \downarrow & & \downarrow \Phi_{(d)} \\ H_{(d)}^1(H, \mathbb{R}) & \xlongequal{\quad} & H^1(H, \text{Pol}_{d-1}(H)) \longrightarrow H^1(H, H_{(d-1)}^1(H, \mathbb{R})) \end{array}$$

where $(\Phi_{(d)} \xi)(\bar{h}) = \Psi_{(d-1)}^1(\xi(\psi_1(\bar{h})))$, $\bar{h} \in H/H_2$.

The proof, split into several parts, takes up the remainder of the present section.

Proof of part (i). Fix G, H and Ω as in the statement. Let (Ω, μ) be a uniform, ergodic (G, H) -coupling. For convenience we will normalize measures such that μ_X, μ_Y are probability measures.

Applying the Reciprocity Theorem. Fix for the moment $d, n \in \mathbb{N}$. By the Reciprocity Theorem 6.4 and the assumption that G has property H_T^0 , we have a commutative diagram

$$\begin{array}{ccc} H_{(d)}^n(G, \mathbb{R}) & \xrightarrow{\quad} & \underline{H}_{(d)}^n(G, L^2 X) \\ \downarrow \cong & & \downarrow \cong \\ H^n(G, \text{Pol}_{d-1}(G)) & \xrightarrow{\cong} & \underline{H}^n(G, L^2 X \otimes \text{Pol}_{d-1}(G)) \end{array}$$

whence the top horizontal arrow is an isomorphism. Recall that the bottom horizontal arrow is an isomorphism cf. Corollary 5.7. Thus we get, again by the Reciprocity theorem, an isomorphism $I: H_{(d)}^n(G, \mathbb{R}) \rightarrow \underline{H}^n(H, L^2(Y, \text{Pol}_{d-1}(G)))$, where we recall that $L^2(Y, \text{Pol}_{d-1}(G))$

is an H -module with action $(h.\xi)(y)(g) = \xi(h^{-1}.y)(\tilde{\omega}_G(h, y)^{-1} \cdot g)$, via the following commutative diagram:

$$\begin{array}{ccc} H_{(d)}^n(G, \mathbb{R}) & \xrightarrow[\cong]{I} & \underline{H}^n(H, L^2(Y, \text{Pol}_{d-1}(G))) \\ & \searrow \cong & \nearrow \cong \\ & \underline{H}_{(d)}^n(G, L^2 X) & \end{array}$$

To prove part (i) we will construct an isomorphism

$$\underline{H}^n(H, L^2(Y, \text{Pol}_{d-1}(G))) \xrightarrow{\Phi} H^n(H, \text{Pol}_{d-1}(H)) ,$$

Recall that by Proposition 6.8 the right-hand side is isomorphic to $H_{(d)}^n(H, \mathbb{R})$.

The map χ . For $\xi \in L^2(Y, \text{Pol}_{d-1}(G))$ consider the function $\chi(\xi)$ on H given by

$$(9.2) \quad (\chi\xi)(h) = \int_Y \xi(y)(\tilde{\omega}_G(h, y)) d\mu_Y(y).$$

Using the cocycle identity and that the action of H on Y preserves the measure, it is easy to see that (9.2) defines an H -equivariant linear map $\chi: L^2(Y, \text{Pol}_{d-1}(G)) \rightarrow L_{loc}^\infty(H)$. Let $\{\zeta_i\}$ be a linear basis of $\text{Pol}_{d-1}(G)$, recalling that $\text{Pol}_{d-1}(G)$ is finite dimensional by Corollary 6.10. For any compact subset K of G and any $\xi = \sum_i f_i \otimes \zeta_i \in L^2(Y, \text{Pol}_{d-1}(G))$ we have for all $h \in K$

$$\begin{aligned} |(\chi\xi)(h)| &\leq \sum_i \left| \int_Y f_i(y) \cdot \zeta_i(\tilde{\omega}_G(h, y)) d\mu_Y(y) \right| \\ &\leq \sum_i \left(\|\zeta_i|_{\tilde{\omega}_G(K \times Y)}\|_\infty \cdot \int_Y |f_i(y)| d\mu_Y(y) \right), \end{aligned}$$

from which it follows that χ is continuous. In particular, the kernel is a closed (H -invariant) subspace, and every element in the image is a continuous vector, whence represented by a continuous function. In summary, χ is a morphism $\chi: L^2(Y, \text{Pol}_{d-1}(G)) \rightarrow C(H, \mathbb{R})$ of continuous Fréchet H -modules.

A module for book keeping. Presently we consider the H -module $\mathcal{H} := \chi(L^2(Y, \text{Pol}_{d-1}(G))) \subseteq C(H, \mathbb{R})$, endowed with the quotient topology. (Which might, a priori, differ from the subspace topology.) For brevity we will denote $\mathcal{L} := L^2(Y, \text{Pol}_{d-1}(G))$ and consider on it the filtration $0 \leq \mathcal{L}_1 \leq \dots \leq \mathcal{L}_d$ where $\mathcal{L}_i := L^2(Y, \text{Pol}_{i-1}(G))$, $i = 1, \dots, d$ and by convention $\mathcal{L}_0 := 0$. Observe that in this way \mathcal{L} is a continuous poly-Hilbert H -module. Further, we have for all i that $\mathcal{L}_i/\mathcal{L}_{i-1} \cong L^2 Y \otimes H_{(i-1)}^1(G, \mathbb{R})$ so that, by ergodicity, $(\mathcal{L}_i/\mathcal{L}_{i-1})^H \cong H_{(i-1)}^1(G, \mathbb{R})$. Finally, we define a filtration of \mathcal{H} by pushing forward that on \mathcal{L} : $\mathcal{H}_i := \chi(\mathcal{L}_i)$, $i = 0, \dots, d$; by construction each quotient $\mathcal{H}_i/\mathcal{H}_{i-1}$ is a continuous, unitary Hilbert H -module, whence \mathcal{H} with this filtration is a continuous poly-Hilbert H module as well.

We claim that $(\mathcal{H}_i/\mathcal{H}_{i-1})^H = \bar{\chi}(\mathbb{R} \otimes H_{(i-1)}^1(G, \mathbb{R}))$, where $\bar{\chi}$ denotes the induced map $\mathcal{L}_i/\mathcal{L}_{i-1} \rightarrow \mathcal{H}_i/\mathcal{H}_{i-1}$. Indeed for each i we have $\mathcal{L}_i/\mathcal{L}_{i-1} \cong \mathcal{H}_i/\mathcal{H}_{i-1} \oplus \ker \bar{\chi}$ from which the claim follows immediately.

The evaluation map. As we observed above, \mathcal{H} embeds continuously as an invariant subspace of $C(H, \mathbb{R})$. In particular there is a continuous linear map $\text{ev}: \mathcal{H} \rightarrow \mathbb{R} : \text{ev}(\xi) = \xi(1)$ induced by evaluation in 1 . It is easy to see that the evaluation map extends continuously to \mathcal{F} : indeed this is clear on $\mathcal{F}_1 = \mathcal{H}_1 = \mathbb{R} \cdot 1_H$. Then in the inductive step of the construction of \mathcal{F} , when considering the intermediate module $\mathcal{F}' \sim \mathcal{F}_{d-1} \times (\mathcal{H}/\mathcal{H}_{d-1})$ we can replace any section $\sigma: \mathcal{H}/\mathcal{H}_{d-1} \rightarrow \mathcal{F}'$ with $\xi \mapsto \sigma(\xi) - (\text{ev}|_{\mathcal{F}_{d-1}} \circ \sigma)(\xi) \cdot 1_H$. Thus for any $\xi \sim (\xi_1, \xi_2) \in \mathcal{F}'$ we can arrange that, by construction, $\text{ev}(\xi) = \text{ev}(\xi_1)$ and the claim now follows.

Now we observe that, given any $\zeta \in \mathcal{F}$, the function $\zeta': h \mapsto \text{ev}(h^{-1} \cdot \zeta)$ is a continuous function on H , that is, the map $\text{ev}_*: \zeta \mapsto \zeta'$ is a morphism $\text{ev}_*: \mathcal{F} \rightarrow C(H, \mathbb{R})$ of H -modules.

Proof proper. With the initial setup now in place we can state the theorem as we actually prove it. Let \mathcal{F} be an H_T^0 -completion of \mathcal{H} . We claim that with hypotheses as in the Theorem, we have:

- (a) The evaluation map restricts to an isomorphism $\text{ev}_*: \mathcal{F}^{H(d)} \xrightarrow{\cong} \text{Pol}_{d-1}(H)$,
- (b) the map $\underline{\chi}_*: \underline{H}^n(H, \mathcal{L}) \rightarrow H^n(H, \mathcal{F})$ is an isomorphism for every $n \in \mathbb{N}$,
- (c) and $\Psi_{(d)}^n$ exists as claimed for all $n \in \mathbb{N}$.

The proof proceeds by induction on d . Note that the $d = 1$ case is (essentially) due to Shalom [30], and follows directly by property H_T^0 and the Reciprocity Theorem 6.4. Fix $d > 1$. If $H^1(H, \mathbb{R}) = 0$, then by the $d = 1$ case we have $H^1(G, \mathbb{R}) = 0$, from which we get $H_{(d)}^1(G, \mathbb{R}) = H_{(d)}^1(H, \mathbb{R}) = 0$ for all $d \in \mathbb{N}$; whence in this case there is nothing to prove, and so we shall suppose that $H^1(H, \mathbb{R}) > 0$.

Locating polynomials. Presently we shall prove part of the inductive step for item (a) above, namely that the evaluation map $\text{ev}_*: \mathcal{F}^{H(d)} \rightarrow \text{Pol}_{d-1}(H)$ is *surjective*. Let $\eta \in \text{Pol}_{d-1}(H)$ and let $\xi: H \rightarrow \mathcal{F}_{d-1}^{H(d-1)}$ be the inhomogeneous 1-cocycle given by $\xi(h) = (\text{ev}_*)^{-1}(\delta_h \eta)$. We need to show that ξ is an *inner* cocycle into \mathcal{F} , that is, that it vanishes in $H^1(H, \mathcal{F})$. By induction and Theorem 5.15 we have isomorphisms

$$H^1(H, \text{Pol}_{d-2}(H)) \xrightarrow{\cong} H^1(H, \mathcal{F}_{d-1}) \xleftarrow{\cong} H^1(G, \text{Pol}_{d-2}(G)) ,$$

whence we conclude that there is a $\xi_0 \in H^1(G, \text{Pol}_{d-2}(G))$ with image ξ . Since ξ_0 is inner (in $H^1(G, \text{Pol}_{d-1}(G))$ whence) in $H^1(G, L^2 X \otimes \text{Pol}_{d-1}(G))$, ξ is inner in $H^1(H, \mathcal{F})$. This proves surjectivity.

Furthermore, this argument shows that, denoting by $\mathcal{Q}' := (\text{ev}_*)^{-1}(\text{Pol}_{d-2})$ and fixing a set of representatives $\{\eta_i\} \subseteq \text{Pol}_{d-1}(H)$ we can arrange that the pre-images $\{\xi_i\}$ satisfy $\partial_h \xi_i \in \mathcal{Q}'$ for all $h \in H$. Thus if we let \mathcal{Q} be the subspace of $\mathcal{F}^{H(d)}$ spanned by \mathcal{Q}' and the ξ_i we conclude readily that (\mathcal{Q} is an H -module and) $\text{ev}_*: \mathcal{Q} \rightarrow \text{Pol}_{d-1}(H)$ is an isomorphism.

Final steps. Let \mathcal{E} be an H_T^0 -completion of \mathcal{L} , and observe that we may construct \mathcal{E}, \mathcal{F} such that χ extends continuously to $\chi: \mathcal{E} \rightarrow \mathcal{F}$. By induction on i , using the long exact sequence, the induced map $\chi_*: H^n(H, \mathcal{E}_{i-1}) \rightarrow H^n(H, \mathcal{F}_{i-1})$ is an isomorphism for each $i = 1, \dots, d$.

We show now by induction on d that the map $\underline{H}^n(H, \mathcal{L}) \rightarrow H^n(H, \mathcal{E})$ is an isomorphism. To see that, observe that the complex, induced from the long exact sequence in continuous cohomology,

$$(9.3) \quad \dots \longrightarrow \underline{H}^{n-1}(H, \mathcal{L}/\mathcal{L}_{d-1}) \xrightarrow{\delta} \underline{H}^n(H, \mathcal{L}_{d-1}) \longrightarrow \underline{H}^n(H, \mathcal{L}) \longrightarrow \underline{H}^n(H, \mathcal{L}/\mathcal{L}_{d-1}) \longrightarrow \dots$$

is exact. (Note that this is not automatic in general.) Indeed, by the proof of the Reciprocity Theorem 6.4, there is a cochain isomorphism of complexes, where we denote for brevity $\mathcal{K} = L^2X \otimes \text{Pol}_{d-1}(G)$ with the obvious composition series $(\mathcal{K}_i)_i$,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{n-1}(H, \mathcal{L}/\mathcal{L}_{d-1}) & \xrightarrow{\delta} & H^n(H, \mathcal{L}_{d-1}) & \longrightarrow & H^n(H, \mathcal{L}) \longrightarrow H^n(H, \mathcal{L}/\mathcal{L}_{d-1}) \longrightarrow \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \cdots & \longrightarrow & H^{n-1}(G, \mathcal{K}/\mathcal{K}_{d-1}) & \xrightarrow{\delta} & H^n(G, \mathcal{K}_{d-1}) & \longrightarrow & H^n(G, \mathcal{K}) \longrightarrow H^n(G, \mathcal{K}/\mathcal{K}_{d-1}) \longrightarrow \cdots \end{array}$$

Since the downward arrows are isomorphisms, they induce also isomorphisms on the level of reduced cohomology, where the bottom row is then, by property H_T^0 and naturality of the long exact sequence, precisely isomorphic to the long exact sequence

$$\begin{array}{c} H^{n-1}(G, H_{(d-1)}^1(G, \mathbb{R})) \longleftarrow \cdots \\ \downarrow \delta \\ H^n(G, \text{Pol}_{d-2}(G)) \longrightarrow H^n(G, \text{Pol}_{d-1}(G)) \longrightarrow H^n(G, H_{(d-1)}^1(G, \mathbb{R})) \longrightarrow \cdots \end{array},$$

from which the observation follows, and hence, by induction and naturality of the long exact sequence in continuous cohomology, that $\underline{H}^n(H, \mathcal{L}) \rightarrow H^n(H, \mathcal{E})$ is an isomorphism. It follows that $\chi_*: \underline{H}^n(H, \mathcal{L}) \rightarrow H^n(H, \mathcal{F})$ is surjective. By what we showed in the step 'Locating polynomials', we can then deduce (a): observe that the map $H^1(H, \mathcal{Q}) \rightarrow H^1(H, \mathcal{F})$ is injective (since $\mathcal{F}^{H(d+1)} = \mathcal{F}^{H(d)}$), whence we have the following diagram:

$$(9.4) \quad H^1(H, \text{Pol}_{d-1}(H)) \cong H^1(H, \mathcal{Q}) \hookrightarrow H^1(H, \mathcal{F}) \leftarrow \underline{H}^1(H, \mathcal{L}) \cong H^1(G, \text{Pol}_{d-1}(G)),$$

by which we conclude that

$$(9.5) \quad \dim_{\mathbb{R}} H^1(H, \text{Pol}_{d-1}(H)) \leq \dim_{\mathbb{R}} H^1(G, \text{Pol}_{d-1}(G)).$$

By symmetry this is then an equality, whence in particular we also get that $\dim_{\mathbb{R}} H^1(H, \mathcal{Q}) = \dim_{\mathbb{R}} H^1(H, \mathcal{F})$. To finish the inductive step for (a), we need to deduce that $\mathcal{Q} = \mathcal{F}^{H(d)}$, which is equivalent, since $\dim_{\mathbb{R}} \text{Pol}_{d-1}(G) = \dim_{\mathbb{R}} \text{Pol}_{d-1}(H)$ by the induction hypotheses, to showing that $(\text{ev}_*)|_{\mathcal{F}^{H(d)}}$ is injective, whence an isomorphism. Inductively $\mathcal{F}_{d-1}^{H(d-1)} \subseteq \mathcal{Q}$ so that, by the construction of \mathcal{F} (namely the fact that $(\mathcal{F}_i/\mathcal{F}_{i-1})^{H(k)} = (\mathcal{F}_i/\mathcal{F}_{i-1})^H, k \geq 1$), we conclude that for every $\xi \in \mathcal{F}^{H(d)}$ and all $h \in H$, we have $\partial_h \xi \in \mathcal{Q}^{H(d-1)}$ on which $(\text{ev}_*)|_{\mathcal{Q}^{H(d-1)}}$ is an isomorphism. Thus if $\text{ev}_*(\xi) = 0$ for some $\xi \in \mathcal{F}^{H(d)} \setminus \mathcal{Q}$ it follows that $\xi \in \mathcal{F}^H$ and that ξ maps to a non-zero element in $\mathcal{F}/\mathcal{F}_{d-1}$.

Let $\zeta: H \rightarrow \mathcal{F}$ be a non-trivial continuous homomorphism with image contained in $\mathbb{R} \cdot \xi$. Then for any inhomogeneous 1-cocycle $\eta: H \rightarrow \mathcal{Q}$ which is not inner, that is, whose range has non-zero projection on $\mathcal{Q}/\mathcal{Q}^{H(d-1)}$, we observe that $\eta - \zeta$ likewise hits an element in $\mathcal{F}^{H(d)} \setminus \mathcal{F}^{H(d-1)}$ and so cannot be inner since $\mathcal{F}^{H(d+1)} = \mathcal{F}^{H(d)}$ by construction of \mathcal{F} . This contradicts the fact that equality holds in (9.5).

Thus we have completed the inductive step for part (a). Part (b) follows immediately by (9.4) since we observed that the two spaces on either extreme have the same dimension whence the surjection in the diagram must be an isomorphism. Finally, by Theorem 5.15 and (a) we have

$$H^n(H, \text{Pol}_{d-1}(H)) \cong H^n(H, \mathcal{F}),$$

which proves (c). \square

Intermezzo. We keep the notations from the proof of part (i). Consider an arbitrary submodule \mathcal{A} of $C(H, \mathbb{R})$. Observe that given any $\xi \in Z^1(G, \mathcal{A})$, it is easy to see that ξ is inner if and only if the function $h \mapsto \text{ev}(\xi(h^{-1}))$ is in \mathcal{A} . Indeed in this case, denoting $f := h \mapsto \text{ev}(\xi(h^{-1})) \in \mathcal{A}$ we have

$$\begin{aligned} (d^1 f)(h)(k) &= ((h - \mathbb{1}).f)(k) \\ &= f(h^{-1}k) - f(k) \\ &= \xi(k^{-1}h)(\mathbb{1}) - \xi(k^{-1})(\mathbb{1}) = \xi(h)(k). \end{aligned}$$

We want to note an analogous observation in \mathcal{F} : Let $\mathcal{F}_{\mathcal{H}}$ denote the H -invariant subspace of \mathcal{F} consisting of elements ζ such that there is a $d \in \mathbb{N}_0$ such that for all h_1, \dots, h_d we have $(h_1 - \mathbb{1}) \cdots (h_d - \mathbb{1}).\zeta \in \mathcal{H}$ for all $h_1, \dots, h_d \in H$. Then it is easy, using the observation that $\mathcal{F}^H = \mathcal{H}^H$, to see that ev_* is injective on $\mathcal{F}_{\mathcal{H}}$, and that a cocycle $\xi \in Z^1(H, \mathcal{H})$ is inner in $H^1(H, \mathcal{F})$ if and only if $h \mapsto \xi(h)(\mathbb{1})$ is in the image $\text{ev}_*(\mathcal{F}_{\mathcal{H}}) \subseteq C(H, \mathbb{R})$.

We can now compare the map $\Psi_{(d)}^1$ to the naive map ω_G^* in (9.1). In fact, we observe that ω_G^* is precisely the composition (to put the right-most term in context, recall that $\text{ev} \circ -$ is precisely the map giving the isomorphism $H^1(H, \text{Pol}_{d-1}(H)) \rightarrow \text{Pol}_d(H)/\text{Pol}_{d-1}(H)$, see Proposition 6.8 and the remarks following it)

$$\text{Pol}_d(G) \longrightarrow H_{(d)}^1(G, \mathbb{R}) \xrightarrow{I} H^1(H, L^2(Y, \text{Pol}_{d-1}(G))) \xrightarrow{\chi_*} H^1(H, \mathcal{H}) \xrightarrow{\text{ev} \circ -} C(H, \mathbb{R}).$$

By part (i) of the theorem it follows that we can describe the image of $\xi \in \text{Pol}_d(G)$ under the map $\Psi_{(d)}^1: \text{Pol}_d(G) \rightarrow \text{Pol}_d(H)/\text{Pol}_{d-1}(H)$ as the unique (modulo $\text{Pol}_{d-1}(H)$) polynomial map $\eta \in \text{Pol}_d(H)$ such that $\eta - \omega_G^*(\xi) \in \text{ev}_*(\mathcal{F}_{\mathcal{H}})$.

Proof of part (ii). Fix a $d > 1$. We keep notation from the proof of part (i). Denote by $P: \mathcal{E} \rightarrow H_{(d-1)}^1(G, \mathbb{R})$ (where we recall from the proof of part (i) that \mathcal{E} is the H_T^0 -completion of $\mathcal{L} := L^2(Y, \text{Pol}_{d-1}(G))$) the projection given by the following composition, where we recall from Section 5 that $\mathcal{E}/\mathcal{E}_{d-1}$ is a direct product of H -modules, $(\mathcal{E}/\mathcal{E}_{d-1})^H = H_{(d-1)}^1(G, \mathbb{R})$ being one of them,

$$\mathcal{E} \longrightarrow \mathcal{E}/\mathcal{E}_{d-1} \longrightarrow (\mathcal{E}/\mathcal{E}_{d-1})^H$$

and denote similarly by Q the projection

$$\mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_{d-1} \longrightarrow (\mathcal{F}/\mathcal{F}_{d-1})^H \xrightarrow{\cong} H_{(d-1)}^1(H, \mathbb{R}),$$

with the final arrow being the map induced by evaluation, as in the proof of part (i). Observe that since $\ker \text{ev}_* \subseteq \ker Q$, we get an induced projection $\text{ev}_*(\mathcal{F}) \rightarrow H_{(d-1)}^1(H, \mathbb{R})$, which we also denote Q .

Step I. We first claim that there is a commutative diagram

$$\begin{array}{ccc} \underline{H}^1(H, \mathcal{L}) & \xrightarrow{P_*} & H^1(H, H_{(d-1)}^1(G, \mathbb{R})) \\ \chi_* \downarrow \cong & & \downarrow (\Psi_{(d-1)}^1)_* \\ H^1(H, \mathcal{F}) & \xrightarrow{Q_*} & H^1(H, H_{(d-1)}^1(H, \mathbb{R})) \end{array}$$

Let $\xi \in Z^1(H, \mathcal{L})$ and $\bar{h} \in H/H_2$ be given. Since the value of $(Q_* \circ \chi_*)(\xi)(\bar{h})$ is independent of the choice of ξ up to the closure of the space $B^1(H, \mathcal{L})$ of coboundaries, we may suppose that $\xi(\bar{h}) \in \text{Pol}_{d-1}(G) + \mathcal{L}_{d-1}$, say $\xi(\bar{h}) = \xi_0 + \eta$; indeed, consider the pre-image $\mathcal{L}' \leq \mathcal{L}$ of $(\mathcal{L}/\mathcal{L}_{d-1})^H$ under the canonical projection. (I.e. $\mathcal{L}' := P^{-1}((\mathcal{E}/\mathcal{E}_{d-1})^H) \cap \mathcal{L}$, if you will.) As we already noted the exactness of the complex in (9.3), it follows that the map in reduced cohomology $\underline{H}^1(H, \mathcal{L}') \rightarrow \underline{H}^1(H, \mathcal{L})$ induced by the embedding $\mathcal{L}' \leq \mathcal{L}$, is surjective.

Then we have

$$(Q_* \circ \chi_*)(\xi)(\bar{h}) = Q \left(k \mapsto \int_Y \xi_0(\tilde{\omega}_G(k, y)) d\mu_Y(y) \right) + (Q_* \circ \chi_*)(\eta)(\bar{h}).$$

But $(Q_* \circ \chi_*)(\eta)(\bar{h}) = 0$ since $\chi_*(\eta)$ vanishes in cohomology, η being in \mathcal{L}_{d-1} whence $\chi_*(\eta) \in \mathcal{H}_{d-1}$. It remains to show that

$$(9.6) \quad Q \left(k \mapsto \int_Y \xi_0(\tilde{\omega}_G(k, y)) d\mu_Y(y) \right) = \Psi_{(d-1)}^1(\xi_0).$$

Since the argument in Q is just $k \mapsto \int_Y \xi_0(\tilde{\omega}_G(k, y)) d\mu_Y(y) = \omega_G^*(\xi_0)$, this follows from the observation in the intermezzo that there is some $f \in \text{ev}_*((\mathcal{F}_{d-1})_{\mathcal{H}_{d-1}})$ and a representative $\eta \in \text{Pol}_{d-1}(H)$ of $\Psi_{(d-1)}^1(\xi_0)$, such that $\eta - \omega_G^*(\xi_0) = f$. Applying Q to both sides of this equality gives (9.6). This finishes Step I.

Step II. To complete the proof we show that there is a commutative diagram

$$\begin{array}{ccc} H^1(G, \text{Pol}_{d-1}(G)) & \longrightarrow & H^1(G, H_{(d-1)}^1(G, \mathbb{R})) \\ \downarrow I \cong & & \downarrow \psi_1^* \\ \underline{H}^1(H, \mathcal{L}) & \xrightarrow{P_*} & H^1(H, H_{(d-1)}^1(G, \mathbb{R})) \end{array}$$

To see this we consider the commutative diagram

$$\begin{array}{ccc} H^1(H, \mathcal{L}) & \xrightarrow{P_*} & H^1(H, H_{(d-1)}^1(G, \mathbb{R})) \\ & \searrow p_* & \nearrow (f_Y -) \\ & H^1(H, L^2(Y, H_{(d-1)}^1(G, \mathbb{R}))) & \end{array}$$

where p is the projection $p: \mathcal{L} \rightarrow \mathcal{L}/\mathcal{L}_{d-1} \cong L^2(Y, H_{(d-1)}^1(G, \mathbb{R}))$. Then we compute for $\xi \in Z^1(G, \text{Pol}_{d-1}(G))$:

$$\begin{aligned} (P_* \circ I)(\xi)(h) &= \left(\left(\int_Y - \right) \circ p_* \circ I \right) (\xi)(h) \\ &= \int_Y p(I(\xi)(h))(y) d\mu_Y(y) \\ &= \int_Y r(I(\xi)(h)(y)) d\mu_Y(y), \end{aligned}$$

where r is the projection $\text{Pol}_{d-1}(G) \rightarrow H_{(d-1)}^1(G, \mathbb{R})$. Thus we continue:

$$\int_Y r(I(\xi)(h)(y)) d\mu_Y(y) = \int_Y (r \circ \xi)(\tilde{\omega}_G(h, y)) d\mu_Y(y).$$

On the right-hand side $(r \circ \xi): G \rightarrow H_{(d-1)}^1(G, \mathbb{R})$ is a homomorphism into a finite dimensional real vector space, and so it follows by the definition of ψ_1 that, denoting by \bar{h} the image of h in H/H_2

$$\int_Y (r \circ \xi)(\tilde{\omega}_G(h, y)) d\mu_Y(y) = (r \circ \xi)(\psi_1(\bar{h})) = r_*(\xi)(\psi(\bar{h})),$$

as had to be shown. This finishes the proof of step II and, putting the two steps together, the proof of part (ii) of Theorem 9.2. \square

10. PROOF OF THEOREM B

Below we will need the following observation. Fix a Mal'cev group G and a $d \in \mathbb{N}$. By the remarks following Proposition 6.8 and using Lemma 7.12, every cocycle $\xi \in Z^1(G, \text{Pol}_{d-1}(G))$ takes values in $\mathbb{R} \cdot \mathbf{1}_G$ on G_d , whence it follows that $H^1(G, \text{Pol}_{d-1}(G)/\mathbb{R} \cdot \mathbf{1}_G) = H^1(G/G_d, \text{Pol}_{d-1}(G)/\mathbb{R} \cdot \mathbf{1}_G)$. Thus if H is a Mal'cev group as well and $\psi: H/H_d \rightarrow G/G_d$ is a homomorphism, we get an induced homomorphism $\psi_\star: H^1(G, \text{Pol}_{d-1}(G)/\mathbb{R} \cdot \mathbf{1}_G) \rightarrow H^1(H, \text{Pol}_{d-1}(H)/\mathbb{R} \cdot \mathbf{1}_H)$, given by

$$\psi_\star(\xi)(\bar{h}) = \psi^*(\xi)(\psi(\bar{h})), \quad \xi \in \text{Pol}_{d-1}(G), \bar{h} \in H/H_d.$$

Remark 10.1. In the proof of Theorem B below we construct, as indicated earlier, an isomorphism using, essentially, a kind of higher order duality argument. Let us briefly indicate an idea which is the same but different. Suppose that G is a csc nilpotent Lie group of class d , such that $G_d \cong \mathbb{R}$. Let z be the element in $G_d \leq G$ corresponding to $1 \in \mathbb{R}$ and extend this to some Mal'cev basis of G . Consider the short exact sequence of G -modules

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Pol}_{d-1}(G) \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q} := \text{Pol}_{d-1}(G)/\mathbb{R} \cdot \mathbf{1}_G$. Since G_d acts trivially on each term, we may also consider this as a short exact sequence of G/G_d -modules, and thus we get two long exact sequences in cohomology as in the following diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^1(G, \text{Pol}_{d-1}(G)) & \longrightarrow & H^1(G, \mathcal{Q}) & \xrightarrow{\delta} & H^2(G, \mathbb{R}) \longrightarrow \cdots, \\ & & & & \parallel & \swarrow & \\ \cdots & \longrightarrow & H^1(G/G_d, \text{Pol}_{d-1}(G)) & \longrightarrow & H^1(G/G_d, \mathcal{Q}) & \xrightarrow{\bar{\delta}} & H^2(G/G_d, \mathbb{R}) \longrightarrow \cdots \end{array}$$

where the middle vertical isomorphism is the one observed just above. Then it is not hard to convince oneself that the map $\varphi: H^1(G, \text{Pol}_{d-1}(G)) \rightarrow H^2(G/G_d, \mathbb{R})$ maps ζ_z to the cohomology class in $H^2(G/G_d, \mathbb{R})$ representing G as a central extension of G/G_d .

Back to the proof of Theorem B

Lemma 10.2. *Let G, H be csc nilpotent Lie groups with $\text{cl}(G) = \text{cl}(H) =: d$. Suppose that $\psi: H/H_d \rightarrow G/G_d$ is a continuous group homomorphism and that $\Psi: H_{(d)}^1(G, \mathbb{R}) \rightarrow$*

$H_{(d)}^1(H, \mathbb{R})$ is a linear map such that the following diagram commutes:

$$(10.1) \quad \begin{array}{ccccc} & & H^1(G, H_{(d-1)}^1(G, \mathbb{R})) & & \\ & \nearrow & \downarrow \psi_\star & \nwarrow & \\ H^1(G, \text{Pol}_{d-1}(G)) & \xrightarrow{\pi_*} & & \xrightarrow{\pi_*} & H^1(G, \text{Pol}_{d-1}(G)/\mathbb{R}) \\ \downarrow \Psi & & \downarrow & & \downarrow \psi_\star \\ & \nearrow & H^1(H, H_{(d-1)}^1(H, \mathbb{R})) & \nwarrow & \\ H^1(H, \text{Pol}_{d-1}(H)) & \xrightarrow{\pi_*} & & \xrightarrow{\pi_*} & H^1(H, \text{Pol}_{d-1}(H)/\mathbb{R}) \end{array}$$

Then ψ extends (uniquely) to a homomorphism $\psi': H \rightarrow G$ such that ψ' induces ψ on the level of quotients, and such that $(\psi')^*: H_{(d)}^1(G, \mathbb{R}) \rightarrow H_{(d)}^1(H, \mathbb{R})$ coincides with Ψ .

Proof. Let $\{g_{i,j} \mid (i,j) \in \mathbf{B}_{\mathbf{rk}(G)}\}$ and $\{h_{i,j} \mid (i,j) \in \mathbf{B}_{\mathbf{rk}(H)}\}$ be Mal'cev bases of G respectively H , and denote by

$$\sigma: G/G_d \rightarrow G: \prod_{(i,j) \in \mathbf{B}_{\mathbf{rk}(G/G_d)}} \bar{g}_{i,j} \mapsto \prod_{(i,j) \in \mathbf{B}_{\mathbf{rk}(G/G_d)}} g_{i,j}$$

the section of $G \rightarrow G/G_d$ induced by the choice of Mal'cev coordinates. Next we claim that for every polynomial map $\xi \in \text{Pol}_d(G)$ (in particular every ξ of the form $\xi = \zeta_{g_{d,j}}$), there is a polynomial map $\eta \in \text{Pol}_d(H)$ such that $\psi^*(\partial_{(\sigma \circ \psi)(\bar{h})} \xi) - \partial_h \eta$ is constant for every $h \in H$, say equal $c(h, \xi)$. Indeed, taking the cocycle representative $\xi_0: g \mapsto \partial_g \xi$ corresponding to ξ , cf. the remarks following Proposition 6.8, we get from the foremost commutative rectangle in (10.1) that $\zeta: h \mapsto \psi^*((\pi_* \xi_0)(\psi(\bar{h}))) - \pi(\Psi(\xi_0)(h))$ is an inner cocycle in $H^1(H, \text{Pol}_{d-1}(H)/\mathbb{R})$. Then since $\pi_*: B^1(H, \text{Pol}_{d-1}(H)) \rightarrow B^1(H, \text{Pol}_{d-1}(H)/\mathbb{R})$ is surjective, we can find $\eta' \in B^1(H, \text{Pol}_{d-1}(H))$ such that $\zeta - \pi_*(\eta') = 0$ and so taking η to be the (unital) polynomial map such that $\partial_h \eta = \Psi(\xi_0)(h) + \eta'(h)$, $h \in H$ we have, in $\text{Pol}_{d-1}(H)/\mathbb{R}$,

$$\psi^*((\pi_* \xi_0)(\psi(\bar{h}))) = \pi(\partial_h \eta), \quad h \in H.$$

The claim follows since $\pi(\xi_0(\psi(h))) = \pi(\partial_{(\sigma \circ \psi)(h)} \xi)$, $h \in H/H_d$, by Lemma 7.12. Now define a map $\psi': \{h_{1,i} \mid i = 1, \dots, \text{rk}(H)\} \rightarrow G$ by

$$\psi'(h_{1,i}) = (\sigma \circ \psi)(h_{1,i}) \cdot \prod_j g_{d,j}^{c(h_{1,i}, j)}.$$

It follows by the above that for every unital $\xi \in \text{Pol}_d(G)$, there is a unique unital (by Lemma 7.10) $\eta \in \text{Pol}_d(H)$ such that

$$(10.2) \quad \partial_{h_{1,i}} \eta = \psi^*(\partial_{\psi'(h_{1,i})} \xi), \quad i = 1, \dots, \text{rk}(H).$$

Indeed, for $\xi \in \text{Pol}_d(G/G_d)$ this is true by the induction hypothesis, and for $\xi = \zeta_{g_{d,j}}$ it is now true by construction, since $\partial_{\psi'(h_{1,i})} \zeta_{g_{d,j}} = \partial_{(\sigma \circ \psi)(\bar{h}_{1,i})} \zeta_{g_{d,j}} - c(h_{1,i}, \zeta_{g_{d,j}})$. Thus (10.2) follows from Lemma 7.7 and linearity of the differential and ψ^* . We denote the induced strongly unital map $\Psi'_0: \text{Pol}_d(G) \rightarrow \text{Pol}_d(H): \xi \mapsto \eta$ and denote $\Psi' := \text{inv} \circ \Psi'_0 \circ \text{inv}$. By construction, the restriction of Ψ' to $\text{Pol}_d(G/G_d)$ coincides with $\psi^* = \text{inv} \circ \psi^* \circ \text{inv}$, and

the composition $\text{Pol}_d(G) \rightarrow \text{Pol}_d(H) \rightarrow H_{(d)}^1(H, \mathbb{R})$ coincides with Ψ (composed with the projection onto $H_{(d)}^1(G, \mathbb{R})$), cf. (10.1). We note that Ψ' satisfies $\Psi'(\xi_0 \cdot \xi_1) = \Psi'(\xi_0) \cdot \Psi'(\xi_1)$ for all $\xi_0, \xi_1 \in \text{Pol}_d(G)$ such that $\deg \xi_0 + \deg \xi_1 \leq d$, which follows since ψ^* (and hence Ψ'_0) does so and the only case not covered by this is the case $\deg \xi_0 = d, \deg \xi_1 = 0$, which is trivial. We also observe that, extending ψ' to $\psi': \{h_{1,i}^t \mid t \in \mathbb{R}, i = 1, \dots, \text{rk}(H)\} \rightarrow G$ by $\psi'(h_{1,i}^t) = \psi'(h_{1,i})^t$, it follows by (10.2), using the cocycle identity for $g \mapsto \partial_g \xi$ (and $h \mapsto \partial_h \eta$), that we have

$$(10.3) \quad \partial_{h_{1,i}^t} \Psi'_0(\xi) = \psi^* (\partial_{\psi(h_{1,i})^t} \xi) = \Psi'_0 (\partial_{\psi'(h_{1,i})^t} \xi), \quad t \in \mathbb{Z}, i = 1, \dots, \text{rk}(H),$$

We will need that this holds as well when replacing ∂ by \mathbb{G} . We observe that for any $t \in \mathbb{Z}$, any $h, k \in \{h_{1,i}^t \mid i, t\}$, we have by induction on $\deg \xi$

$$\begin{aligned} \partial_h (\mathbb{G}_k \Psi'_0(\xi)) &= \mathbb{G}_k (\partial_h \Psi'_0(\xi)) \\ &= \mathbb{G}_k \Psi'_0 (\partial_{\psi'(h)} \xi) \\ &= \Psi'_0 (\mathbb{G}_{\psi'(k)} \partial_{\psi'(h)} \xi), \end{aligned}$$

where we have used that $\partial_{\psi'(h)} \xi \in \text{Pol}_{d-1}(G)$, so that, denoting \bar{k} the projection of k in H/H_d , we have $\mathbb{G}_{\psi(\bar{k})} (\partial_{\psi'(h)} \xi) = \mathbb{G}_{\psi'(k)} (\partial_{\psi'(h)} \xi)$. Thus it follows now, using (10.3), that

$$\begin{aligned} \partial_h (\mathbb{G}_k \Psi'_0(\xi)) &= \Psi'_0 (\partial_{\psi'(h)} (\mathbb{G}_{\psi'(k)} \xi)) \\ &= \partial_h \Psi'_0 (\mathbb{G}_{\psi'(k)} \xi). \end{aligned}$$

Since h could be an arbitrary generator, the claim will follow from Lemma 7.10, given that $(\mathbb{G}_k \Psi'_0(\xi))(\mathbb{1}) = \Psi'_0 (\mathbb{G}_{\psi'(k)} \xi)(\mathbb{1})$; to see that this is indeed the case, we observe that for any $\xi \in \text{Pol}_d(G)$ we have by construction that $(\Psi'_0 \xi)(\mathbb{1}) = \xi(\mathbb{1})$, and thus we compute:

$$\begin{aligned} (\mathbb{G}_k \Psi'_0(\xi))(\mathbb{1}) &= (\partial_{k^{-1}} \Psi'_0(\xi))(\mathbb{1}) \\ &= \Psi'_0 (\partial_{\psi'(k^{-1})} \xi)(\mathbb{1}) \\ &= (\partial_{\psi'(k^{-1})} \xi)(\mathbb{1}) \\ &= \xi(\psi'(k^{-1})^{-1}) - \xi(\mathbb{1}) \\ &= \xi(\psi'(k)) - \xi(\mathbb{1}) = (\mathbb{G}_{\psi'(k)} \xi)(\mathbb{1}), \end{aligned}$$

which proves the claim. Hence it follows, using that $\partial_h \tilde{\xi} = \text{inv}(\mathbb{G}_h \xi)$ and $\mathbb{G}_h \tilde{\xi} = \text{inv}(\partial_h \xi)$, that

$$(10.4) \quad \partial_{h_{1,i}^t} \Psi'(\xi) = \Psi' (\partial_{\psi'(h_{1,i})^t} \xi), \quad t \in \mathbb{Z}, i = 1, \dots, \text{rk}(H)$$

$$(10.5) \quad \mathbb{G}_{h_{1,i}^t} \Psi'(\xi) = \Psi' (\mathbb{G}_{\psi'(h_{1,i})^t} \xi), \quad t \in \mathbb{Z}, i = 1, \dots, \text{rk}(H).$$

Finally, we claim that Ψ' is co-multiplicative in degree d . This follows essentially from the equations (10.4), (10.5) just established, and induction on the degree $\deg \xi$ in the argument in $(\tilde{m} \circ \Psi')(\xi)$, the co-multiplicativity being trivial on constant functions. First note that for any $\xi \in \text{Pol}_d(G)$ and any generator $h_{1,i}$ we get by (10.4) the following computation (where

we use the induction hypothesis in the third equality)

$$\begin{aligned}
\partial_{(h_{1,i}, \mathbb{1})} \tilde{m}^*(\Psi'(\xi)) &= \tilde{m}^*(\partial_{h_{1,i}} \Psi'(\xi)) \\
&= \tilde{m}^*(\Psi'(\partial_{\psi'(h_{1,i})} \xi)) \\
&= (\Psi' \otimes \Psi')(\tilde{m}^*(\partial_{\psi'(h_{1,i})} \xi)) \\
&= (\Psi' \otimes \Psi')(\partial_{(\psi'(h_{1,i}), \mathbb{1})} \tilde{m}^*(\xi)) \\
&= \partial_{(h_{1,i}, \mathbb{1})}(\Psi' \otimes \Psi')(\tilde{m}^*(\xi)).
\end{aligned}$$

Next we need to compute similarly $\partial_{(\mathbb{1}, h_{1,i})} \tilde{m}^*(\Psi'(\xi))$ where we get, entirely as in the computation above that

$$\begin{aligned}
\partial_{(\mathbb{1}, h_{1,i})} \tilde{m}^*(\Psi'(\xi)) &= \tilde{m}^*(\mathbb{G}_{h_{1,i}^{-1}} \Psi'(\xi)) \\
&= \tilde{m}^*(\Psi'(\mathbb{G}_{\psi'(h_{1,i})^{-1}} \xi)) \\
&= (\Psi' \otimes \Psi')(\tilde{m}^*(\mathbb{G}_{\psi'(h_{1,i})^{-1}} \xi)) \\
&= (\Psi' \otimes \Psi')(\partial_{(\mathbb{1}, \psi'(h_{1,i}))} \tilde{m}^*(\xi)) \\
&= \partial_{(\mathbb{1}, h_{1,i})}(\Psi' \otimes \Psi')(\tilde{m}^*(\xi)).
\end{aligned}$$

Finally, since Ψ' is strongly unital and \tilde{m}^* as well, we have $\tilde{m}(\Psi'(\xi))(\mathbb{1}, \mathbb{1}) = (\Psi' \otimes \Psi')(\tilde{m}^*(\xi))(\mathbb{1}, \mathbb{1})$ as well, and so by Lemma 7.10 we conclude that

$$\tilde{m}^*(\Psi'(\xi)) = (\Psi' \otimes \Psi')(\tilde{m}^*(\xi))$$

for all $\xi \in \text{Pol}_d(G) = \text{Pol}_d(G)$, that is, that Ψ' is co-multiplicative in degree d .

Thus Proposition 8.9 (and Remark 8.10) shows that ψ' extends to a homomorphism $\psi': H \rightarrow G$. The fact that $(\psi')^*: H_{(d)}^1(G, \mathbb{R}) \rightarrow H_{(d)}^1(H, \mathbb{R})$ coincides with Ψ follows directly using the middle column of diagram (10.1). \square

We can finally complete the proof of Theorem B. Given G, H and Ω as in the statement, we will construct inductively isomorphisms $\psi_d: H/H_{d+1} \xrightarrow{\cong} G/G_{d+1}$ such that the induced maps $\psi_d^*: H_{(d)}^1(G, \mathbb{R}) \rightarrow H_{(d)}^1(H, \mathbb{R})$ coincide with the isomorphisms $\Psi_{(d)}^1$ of Theorem 9.2. Observe that the existence of ψ_1 is immediate by Theorem 9.2. Given $\psi_1, \dots, \psi_{d-1}$ we proceed as follows.

Fix a Mal'cev basis $\{h_{i,j} \mid (i,j) \in \mathbf{B}_{\text{rk}(H/H_{d+1})}\}$ of H/H_{d+1} and let F be a free nilpotent Lie group of class d , with generators $\{f_{1,j} \mid j = 1, \dots, n\}$, where we denote $n := \text{rk}(H)$. Let $\pi: F \rightarrow H: f_{1,j} \mapsto h_{1,j}$ be the induced, surjective homomorphism of Lie groups. Denote $\tilde{\psi}_{d-1} = \psi_{d-1} \circ \bar{\varphi}$ and $\tilde{\Psi}_{(d)}^1 = \varphi^* \circ \Psi_{(d)}^1$, and consider the following diagram:

$$\begin{array}{ccccc}
& & H^1(G, H^1_{(d-1)}(G, \mathbb{R})) & & \\
& \nearrow & \downarrow (\tilde{\psi}_{d-1})_\star & \nwarrow & \\
H^1(G, \text{Pol}_{d-1}(G)) & \xrightarrow{\quad} & H^1(G, \text{Pol}_{d-1}(G)/\mathbb{R}) & & \\
\downarrow \tilde{\Psi}_{(d)}^1 & & \downarrow (\tilde{\psi}_{d-1})_\star & & \\
& \nearrow & H^1(F, H^1_{(d-1)}(F, \mathbb{R})) & \nwarrow & \\
H^1(F, \text{Pol}_{d-1}(F)) & \xrightarrow{\quad} & H^1(F, \text{Pol}_{d-1}(F)/\mathbb{R}) & & \\
& & \downarrow \iota & &
\end{array}$$

By the induction hypotheses and Theorem 9.2, the two squares in the back are commutative. To see that the front square is commutative we observe that it follows easily from Lemma 7.13 that the arrow labeled ' ι ' in the diagram is injective (in fact an isomorphism), and the commutativity follows directly from this. By the previous lemma we get a homomorphism $\tilde{\psi}: F \rightarrow G$ which induces $\tilde{\psi}_{d-1}$ on the level of quotients (although this is, of course, already clear since F is free nilpotent), and inducing $\tilde{\Psi}_{(d)}^1$ on the level of cohomology (this is the important part). To finish the proof, we will show that $\tilde{\psi}$ factors through H/H_{d+1} . Consider the following diagram:

$$\begin{array}{ccccc}
\text{Hom}(G_d/G_{d+1}, \mathbb{R}) & \xrightarrow{\tilde{\psi}^*} & \text{Hom}(F_d/F_{d+1}, \mathbb{R}) & \xleftarrow{\varphi^*} & \text{Hom}(H_d/H_{d+1}, \mathbb{R}) \\
\downarrow \iota \quad \vdots \quad \pi & & \downarrow \iota \quad \vdots \quad \pi & & \downarrow \iota \quad \vdots \quad \pi \\
& & H^1_{(d)}(F, \mathbb{R}) & & \\
\uparrow \tilde{\Psi}_{(d)}^1 & & \nwarrow \varphi^* & & \\
H^1_{(d)}(G, \mathbb{R}) & \xrightarrow{\Psi_{(d)}^1} & H^1_{(d)}(H, \mathbb{R}) & &
\end{array}$$

where the dashed arrows labeled π are given by choosing a unital representative and restricting. This diagram is commutative by construction, and every solid arrow (that is, every arrow not labeled ' π ') is injective – this is true a priori for every arrow except $\tilde{\psi}^*$, and then follows for this by commutativity of the diagram; it can also be seen directly from the construction of $\tilde{\psi}$. It follows that $\text{im}(\tilde{\psi}^*) = \text{im}(\varphi^*)$, whence that the annihilators coincide as well: $\ker(F_d \rightarrow G_d/G_{d+1}) = \ker(F_d \rightarrow H_d/H_{d+1})$, call it $K \leq F$ and denote $\bar{F} := F/K$. We now have *embeddings* $\tilde{\psi}^*: \text{Pol}_d(G) \hookrightarrow \text{Pol}_d(\bar{F})$ and $\varphi^*: \text{Pol}_d(H) \hookrightarrow \text{Pol}_d(\bar{F})$, and our job is to prove that their images coincide. But this now follows readily from the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(G_d/G_{d+1}, \mathbb{R}) & \longrightarrow & \mathrm{Pol}_d(G) & \longrightarrow & \mathrm{Pol}_d(G/G_d) \longrightarrow 0 \\
& & \downarrow \cong & & \downarrow \tilde{\psi}^* & & \downarrow (\tilde{\psi}_{d-1})^* \\
0 & \longrightarrow & \mathrm{Hom}(\bar{F}_d, \mathbb{R}) & \longrightarrow & \mathrm{Pol}_d(\bar{F}) & \longrightarrow & \mathrm{Pol}_d(\bar{F}/\bar{F}_d) \longrightarrow 0 \\
& & \uparrow \cong & & \uparrow \varphi^* & & \uparrow \varphi^* \\
0 & \longrightarrow & \mathrm{Hom}(H_d/H_{d+1}, \mathbb{R}) & \longrightarrow & \mathrm{Pol}_d(H) & \longrightarrow & \mathrm{Pol}_d(H/H_d) \longrightarrow 0
\end{array}$$

It follows that the map $((\varphi^*)^\dagger)^{-1} \circ \tilde{\psi}^*: \mathrm{Pol}_d(G) \rightarrow \mathrm{Pol}_d(H)$ is a co-multiplicative, strongly unital linear isomorphism whence induces an isomorphism $\psi_d: H/H_{d+1} \rightarrow G/G_{d+1}$ on the level of Lie groups cf. Proposition 8.9 and Remark 8.10. It is easy to see that ψ_d does induce ψ_{d-1} on the level of quotients, and $\Psi_{(d)}^1$ on the level of cohomology. This completes the inductive step and, hence, the proof of Theorem B. \square

Remark 10.3. The proof of Theorem B would proceed more smoothly if we knew a priori, in the inductive step, whether or not the diagram

$$\begin{array}{ccc}
H^1(G, \mathrm{Pol}_{d-1}(G)) & \xrightarrow{\pi_*} & H^1(G, \mathrm{Pol}_{d-1}(G)/\mathbb{R}) \\
\Psi_{(d)}^1 \downarrow & & \downarrow (\psi_{d-1})_\star \\
H^1(H, \mathrm{Pol}_{d-1}(H)) & \xrightarrow{\pi_*} & H^1(H, \mathrm{Pol}_{d-1}(H)/\mathbb{R})
\end{array}$$

is commutative. Proving that it is appears to rely on determining a more explicit description of $\Psi_{(d)}^1$ on the level of polynomial maps, in terms of the map ω_G^* (see Section 9).

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